# Bosonisations and Differentials on Inhomogeneous Quantum Groups

Ryan Kasyfil Aziz

School of Mathematical Sciences

Queen Mary, University of London

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Details of collaboration and publications :

- R. Aziz and S. Majid, Co-double bosonisation and dual bases of $c_q[SL_2]$ and $c_q[SL_3]$, *J. Algebra* **518** (2019) 75–118. [2]
- R. Aziz and S. Majid, Quantum differentials on cross product Hopf algebras, Arxiv: 1904.02662 [math.QA]. [3]

**To my Mom and the memory of my Dad**

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# Abstract

We dualise Majid's double bosonisation to find a construction of coquasitriangular Hopf $B^{\underline{\mathrm{op}}}{>\!\!\!\triangleleft} A {\triangleright\!\!\!<} B^*$ which we call codouble bosonisation, where $B$ is a finite-dimensional braided Hopf algebra living in the category of comodules over coquasitriangular Hopf algebra $A$. We then construct a reduced quantum coordinate algebra $c_q[SL_2]$ at $q$ primitive $n$-th of unity by codouble bosonisation and find new generators for $c_q[SL_2]$ such that their monomials are essentially a dual basis to the standard PBW basis of the reduced Drinfeld-Jimbo quantum enveloping algebra $u_q(sl_2)$. Our methods apply in principle for general $c_q[G]$ as we illustrate for the case of $c_q[SL_3]$ at certain odd roots of unity.

We also introduce a method of finding differential calculi on double cross product $A{\bowtie}H$, biproduct $A{\triangleright\!\!\!<}B$, and bicrossproduct $A{\blacktriangleright\!\!\!\triangleleft}H$ Hopf algebras by constructing their super version. We apply our method to construct the natural differential calculus on the generalised quantum double $D(A,H) = A^{\mathrm{op}}{\bowtie}H$ such that the resulting exterior algebra acts differentiably on $H$, and on the double coquasitriangular Hopf algebras $A{\bowtie}_{\mathcal{R}}A$ such that the resulting exterior algebra acts and coacts differentiably on $A$. We also construct $\Omega(\mathbb{C}_q[GL_2{\triangleright\!\!\!<}\mathbb{C}^2])$ for the quantum group of affine transformation of the plane and $\Omega_\lambda(\mathrm{Poinc}_{1,1})$ for the bicrossproduct Poincaré group in 2 dimensions such that the resulting exterior algebras are strongly bicovariant and coact differentiably on the canonical comodule algebras associated to these inhomogeneous quantum groups.

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# Chapter 1

# Introduction

Research on Hopf algebras has a long history, starting with Hopf [13] in his study of $H$-spaces in algebraic topology. The term 'Hopf algebras' began to be used in late 1960s with the publication of the book of Sweedler [45]. Early examples of Hopf algebras include group algebras $kG$ and enveloping algebras $U(\mathfrak{g})$ associated to complex semisimple Lie algebras $\mathfrak{g}$ and in early times Hopf algebras were mainly used as a tool to unify group and Lie algebra constructions.

More interesting early examples are the Sweedler-Taft algebra [45] $U_q(b_+)$ and $U_q(sl_2)$ [44] since these are not only non-commutative, but also non-cocommutative. They can be viewed as deformations of $U(b_+)$ and $U(sl_2)$ respectively, where $b_+$ is a positive Borel subalgebra of $sl_2$. The theory became particularly important in the late 1980s when Drinfeld [11] and Jimbo [15] independently found a deformation of the enveloping algebra, denoted by $U_q(\mathfrak{g})$, for all complex semisimple lie algebras $\mathfrak{g}$. Moreover, the noncocommutativity of $U_q(\mathfrak{g})$ is controlled by an element $\mathcal{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ if we are working over formal series $\mathbb{C}[[t]]$, which turns out to be a solution of the quantum Yang-Baxter equation. Drinfeld introduced the notion of such quasitriangular Hopf algebras $(H, \mathcal{R})$ with suitable axioms for $\mathcal{R}$. He also introduced the quantum double $D(H)$, at least if $H$ is finite-dimensional, which is nontrivially quasitriangular.

The discovery of quantum groups $U_q(\mathfrak{g})$ opened a new role of Hopf algebras in knot theory and algebra, such as Luzstig's construction of quantum groups by 'divided powers' [16] and their finite-dimensional quotients $u_q(\mathfrak{g})$ at $q$ a primitive $n$-th root of unity [14]. There are also corresponding 'coordinate algebra' $C_q[G]$ which are quotients of a bialgebra $A(R)$ due to Faddeev, Reshetikhin, and Takhtajan [12] in the 1980s, and in principle reduced finite-dimensional quotients $c_q[G]$, best understood for specific cases [9]. Coming from a non-commutative geometry point of view was another class of quantum groups in the 1980s [23], the bicrossproduct Hopf algebras initially as quantisation of classical phase space and later also developed as quantum symmetry of 'non-commutative spacetime' algebras [36].

Many models of quantum geometries and their symmetries were explored in the 1990's. However, in many cases the more relevant structure was not necessarily a Hopf algebra. For example the 2-dimensional quantum-braided plane $\mathbb{C}^2_q$ has relation $yx = qxy$ and primitive coproduct $\Delta x = 1 \otimes x + x \otimes 1$, similarly for $\Delta y$. One can check that $\mathbb{C}^2_q$ in this setting is not a Hopf algebra, but rather a *braided Hopf algebra* [24, 31] as first introduced by Majid. The idea of braided Hopf algebras is to quantise the tensor product of algebras rather than just the algebra itself, i.e. for algebras $B, C$, its tensor product algebra $B \underline{\otimes} C$ has product

$$(b \otimes c)(a \otimes d) = b\Psi(c \otimes d)a,$$

for all $a, b \in B$ and $c, d \in C$, and braiding $\Psi : C \otimes B \to B \otimes C$. Formally, we define a braided Hopf algebra $B$ to be an algebra and coalgebra in a braided monoidal category $\mathcal{C}$, where the unit element is viewed as a morphism $\eta : \underline{1} \to B$ from the unit object, and the product, counit, coproduct and antipode are morphisms such that they have the similar axioms as a Hopf algebra but with the braiding $\Psi$ on $B \otimes B$.

One of the main results in braided Hopf algebras is bosonisation [26], which is a method to construct an ordinary Hopf algebra from a braided Hopf algebra living in the category of (co)modules over a (co)quasitriangular Hopf algebra. Moreover, in [28] Majid introduced *double bosonisation* which associates to each finite-dimensional braided Hopf algebra $B$ living in the category of modules over a quasitriangular Hopf algebra $H$, a new quasitriangular Hopf algebra

$$B^{*\underline{\mathrm{cop}}}\bowtie H \bowtie B =: D_H(B),$$

where the second notation has also been used in the literature in line with the view of this in [29] as the closest one can come to the bosonisation of a 'braided double' of $B$ (the latter does not itself exist in the strictly braided case). Majid also proved that quantum groups $U_q(\mathfrak{g})$ can be constructed inductively by double bosonisation.

In spite of some extensive literature on $U_q(\mathfrak{g})$ or its reduced finite-dimensional case $u_q(\mathfrak{g})$ at primitive $n$-th root of unity, one problem which has been open even for the simplest case of $u_q(sl_2)$ is a description of the dual basis of $c_q[SL_2]$ in terms of the generators and relations. Here $u_q(sl_2)$ has generators $F, K, E$ with the relations of $U_q(sl_2)$ and additionally $E^n = F^n = 0, K^n = 1$ and can be constructed for $n$ odd by double bosonisation as $u_q(sl_2) = (\mathfrak{c}_q^1)^{*\underline{\mathrm{cop}}}\bowtie \mathbb{C}\mathbb{Z}_n \bowtie \mathfrak{c}_q^1$, where $\mathfrak{c}_q^1$ is the reduced braided line. Double bosonisation also gives the PBW basis $\{F^i K^j E^k\}_{0\leq i,j,k<n}$ of $u_q(sl_2)$. The dual Hopf algebra $c_q[SL_2]$ is a quotient of $\mathbb{C}_q[SL_2]$ with its standard matrix entry generators $a, b, c, d$, and the additional relations $a^n = d^n = 1, b^n = c^n = 0$ to give a Hopf algebra extension

$$\mathbb{C}[SL_2] \hookrightarrow \mathbb{C}_q[SL_2] \twoheadrightarrow c_q[SL_2].$$

This $c_q[SL_2]$ has an obvious monomial basis $\{b^i a^j c^k\}$ but its Hopf algebra pairing with the PBW basis of $u_q(sl_2)$ is rather complicated and does not form a dual basis even up to normalisation. Knowing a basis and dual basis is equivalent to knowing the canonical coevaluation element, which has many applications including Hopf algebra Fourier transform.

In this thesis, I solve the dual basis problem for $c_q[SL_2]$ at $q$ a primitive odd root of unity. This is now published in a joint work [2]. My approach to the dual basis problem is to work out the dual version of double bosonisation or *codouble bosonisation* and use this to construct $c_q[SL_2]$ in the dual form

$$B^{\underline{\rm op}}{>\!\!\!\triangleleft\kern-.33em\cdot} A {\cdot\kern-.33em\triangleright\!\!\!<} B^* =: coD_A(B),$$

where each tensor factor pairs with the corresponding factor on the $u_q(sl_2)$ side. This dual version of double bosonisation is in Chapter 3 and is conceptually given by reversing arrows in the original construction, but in practice takes a great deal of care to trace through all the layers of the construction. We will find new generators $x, t, y$ of $c_q[SL_2]$ such that normalised monomials $\{x^i t^j y^k\}$ *are* essentially a dual basis in the sense of being dually paired by

$$\langle x^i t^j y^k, F^{i'} K^{j'} E^{k'}\rangle = \delta_{ii'}\delta_{kk'} q^{jj'} [i]_{q^{-1}}![k]_q!,$$

where $[i]_q$ etc. are $q$-integers.

In general, we can construct $u_q(sl_k) = \mathfrak{c}_q^{k-1}{>\!\!\!\triangleleft\kern-.33em\cdot}\widetilde{u_q(sl_{k-1})}{\cdot\kern-.33em\triangleright\!\!\!<}\mathfrak{c}_q^{k-1}$ and similarly its dual version $c_q[SL_k] = \mathfrak{c}_q^{k-1}{>\!\!\!\triangleleft\kern-.33em\cdot} c_q\widetilde{[SL_{k-1}]}{\cdot\kern-.33em\triangleright\!\!\!<}\mathfrak{c}_q^{k-1}$. We will illustrate this for $u_q(sl_3)$ and $c_q[SL_3]$ for some $n$ in Chapter 4. There are also choices which do not have classical limit as we illustrate in this thesis with $A = \mathbb{C}_q[GL_2]$ not finite-dimensional, $q$ generic and $B = \mathbb{C}_q^{0|2}$ the 'fermionic braided plane' in the category of $A$-comodules. This leads to an exotic but still coquasitriangular version of $\mathbb{C}_q[SL_3]$ with some matrix entries 'fermionic'.

After constructing quantum groups which can be regarded as quantum geometries, one could also study their noncommutative differentials using the tools of Hopf algebras. This is a different approach to the 'Dirac operator' approach by Connes [10], where in our case we start with differential structures on quantum groups or Hopf algebras and associated comodule algebras, expressed in the form of a differential graded algebra (DGA), see for instance [7, 20, 21]. Having at least 1-forms $\Omega^1$ and 2-forms $\Omega^2$, we can construct basic elements of Riemannian geometry such as metrics, connections, curvature and torsion which are all defined algebraically on the DGA.

A fundamental issue here is that in general, there will be many $\Omega^1$ and $\Omega^2$ on a given

Hopf algebra even if we demand left and right translation invariance (i.e., bicovariance). The only general result here remains that of Woronowicz [47] that bicovariant $\Omega^1$ correspond to (say) right ideals of the augmentation ideal stable under the right adjoint coaction. This, however, only translates the problem. If the Hopf algebra is coquasitriangular then a general classification was obtained in [5, 32] in terms of the irreducible corepresentations, but for other types of Hopf algebras there are no such general results. In particular for inhomogeneous quantum groups, there are many ideals and natural differential structure are not usually known.

Here, in Chapter 5 – 7, we provide a different approach from Woronowicz that works for inhomogeneous quantum groups. This work is based on a joint paper [3]. My approach does not classify all calculi but rather selects out a natural calculus with the property that the inhomogeneous quantum group coacts or acts *differentiably* on the canonically associated comodule or module algebra. This is covered in four flavours : double cross product Hopf algebras $A\bowtie H$ where $A$ right-acts on $H$ and simultaneously $H$ left-acts on $A$, double cross coproduct $H\blacktriangleright\!\!\blacktriangleleft A$ where $H$ right-coacts on $A$ and simultaneously $A$ left-coacts on $H$, bicrossproduct $A\blacktriangleright\!\!\triangleleft H$ where $H$ left-acts on $A$ and $A$ right-coacts on $H$ [19, 23], and biproduct or bosonisation $A\triangleright\!\!\!<B$.

We start with the notion of a *strongly bicovariant* exterior algebra $\Omega(A)$ introduced recently in [39]. We recall that a DGA on Hopf algebra $A$ means $\Omega(A)=\oplus_i\Omega^i$ a graded algebra with a graded derivation d with $\mathrm{d}^2=0$, and we say $\Omega$ is an exterior algebra if it is generated by $\Omega^0=A$ and $\Omega^1=A\mathrm{d}A$. This is strongly bicovariant if $\Omega(A)$ is a super-Hopf algebra with coproduct $\Delta_*$ (say) with $\Delta_*|_A=\Delta$ the coproduct of $A$ in degree zero and d is a graded coderivation in the sense

$$\Delta_*\mathrm{d}=(\mathrm{d}\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*$$

where $(-1)^{|\ |}\omega=(-1)^{|\omega|}$ for $\omega$ of degree $|\omega|$. It was shown in [39] that the canonical exterior algebra due to Woronowicz [47] of a first order bicovariant calculus is strongly bicovariant with $\Delta_*|_{\Omega^1} = \Delta_L + \Delta_R$ in terms of the coactions on $\Omega^1$ induced by $\Delta$ (that the Woronowicz construction gives a super-Hopf algebra here was already known [8] while the supercoderivation property was new). Conversely, if $\Omega(A)$ is strongly bicovariant then $\Delta_*|_{\Omega^i} \subset (A \otimes \Omega^i) \oplus \cdots \oplus (\Omega^i \otimes A)$ recovers the left and right coactions on each $\Omega^i$ as the outer components, making $\Omega(A)$ bicovariant as a left and right comodule algebra, but contains a lot more information in the intermediate components. It is natural to add to this the idea that an algebra $B$ with a calculus $\Omega(B)$ has a right coaction $\Delta_R : B \to B \otimes A$ that extends to $\Delta_{R*} : \Omega(B) \to \Omega(B)\underline{\otimes}\Omega(A)$ as a super comodule algebra. Such $\Delta_R$ is said to be *differentiable.* This is used in [7] in a different context from this thesis, namely in the theory of quantum principal bundles and fibrations, but we use this notion to construct $\Omega(B)$ if $\Omega(A)$ is given. We will also need the notion of an $A$ *acting differentiably* on $B$ to make $\Omega(B)$ a super $\Omega(A)$-module algebra which we introduce in Chapter 5.

# Chapter 2

# Preliminaries

We recall the notations and facts about Hopf algebras as can be found in several texts, for example in [1, 14, 45], but we mostly follow [19, 20]. In particular, we focus on the concepts of quasitriangular Hopf algebras as first introduced by Drinfeld [11], and its dual notion of coquasitriangular Hopf algebras. We also recall the notion of Hopf algebras in a braided monoidal category, or braided Hopf algebras [24, 31] and discuss biproduct Hopf algebras or bosonisation [19, 20, 26]. Finally we also recall the definition and some properties of differentials on Hopf algebras, which can be found for example in [7, 21], as well as the less well-known notion of strongly bicovariant calculus [39].

We work over a ground field $k$. By 'vector space' we mean a vector space over $k$. The tensor product of two spaces $V \otimes W$ is understood to be over $k$. All maps here are linear maps.

## 2.1 Hopf algebras

### 2.1.1 Definition and basic properties

Recall that a (unital and associative) algebra $(A,\cdot,\eta)$ means a vector space $A$ equipped with a multiplication as a map $\cdot : A\otimes A\to A$ and a unit as a map $\eta : k\to A$ satisfying

$$\cdot\,(\mathrm{id}\otimes\cdot)=\cdot\,(\cdot\otimes\mathrm{id}),\quad \cdot(\mathrm{id}\otimes\eta)=\mathrm{id}=\cdot(\eta\otimes\mathrm{id}).$$

One can recover the usual unit of $A$ as $1_A=\eta(1_k)$ where $1_k$ is the unit of $k$, and if there is no confusion, we will write 1 for the unit of an algebra.

If $A$ and $B$ are algebras, then a map $f : A\to B$ is an algebra map if it satisfies $f(ab)=f(a)f(b)$ and $f(1)=1$. The tensor product $A\otimes B$ is also an algebra with product and unit

$$\cdot_{A\otimes B}=(\cdot_A\otimes\cdot_B)(\mathrm{id}\otimes\mathrm{flip}\otimes\mathrm{id}),\quad \eta_{A\otimes B}=\eta_A\otimes\eta_B,$$

where flip is a map $B\otimes A\to A\otimes B$ such that $\mathrm{flip}(b\otimes a)=a\otimes b$. It means that the product and unit of $A\otimes B$ can be written explicitly as $(a\otimes b)(c\otimes d)=ab\otimes cd$ and $1_{A\otimes B}=1\otimes 1$. We denote $A^{\mathrm{op}}$ as the opposite algebra of $A$ with product $\cdot_{\mathrm{op}}=\cdot\circ\mathrm{flip}$ and the same unit as $A$. We then have $A$ commutative if and only if $A=A^{\mathrm{op}}$.

One can write the axioms for an algebra in terms of commutative diagram and reverse its arrows to obtain the dual notion of a coalgebra

**Definition 2.1.1.** A (counital and coassociative) *coalgebra* $(C,\Delta,\epsilon)$ means a vector space $C$ equipped with a 'coproduct' map $\Delta : C\to C\otimes C$ and a 'counit' map $\epsilon : C\to k$ such that

$$(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta,\quad (\epsilon\otimes\mathrm{id})\Delta=\mathrm{id}=(\mathrm{id}\otimes\epsilon)\Delta.$$

We use Sweedler notation $\Delta c=\sum c_{(1)}\otimes c_{(2)}$ but when doing calculations and proofs we do not write the sum sign which should be understood. Using this notation, the coassociativity axiom can be written as $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$, and the iterated indices can be renumbered as $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. By applying $\Delta$ repeatedly we have $\Delta^n c = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \otimes c_{(4)} \cdots \otimes c_{(n+1)}$, where $\Delta^n = (\Delta \otimes \mathrm{id})\Delta^{n-1} = (\mathrm{id} \otimes \Delta)\Delta^{n-1}$. We can also write the counity axiom as $\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)})$.

If $C, D$ are coalgebras, then a map $f : C \to D$ is called a coalgebra map if $\Delta_D f = (f \otimes f)\Delta_C$ and $\epsilon_D \circ f = \epsilon_C$. The tensor product $C \otimes D$ is also a coalgebra with coproduct and counit

$$\Delta_{C\otimes D} = (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id})(\Delta_C \otimes \Delta_D), \quad \epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D,$$

which can be written explicitly as $\Delta(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$ and $\epsilon(c \otimes d) = (\epsilon c) \otimes (\epsilon d)$. We denote $C^{\mathrm{cop}}$ as the coopposite coalgebra of $C$ with coproduct $\Delta_{\mathrm{cop}} = \mathrm{flip} \circ \Delta$ and the same counit as $C$. We then have $C$ cocommutative if and only if $C^{\mathrm{cop}} = C$.

**Definition 2.1.2.** A vector space $(B, \cdot, \eta, \Delta, \epsilon)$ is called a *bialgebra* if $(B, \cdot, \eta)$ is an algebra and $(B, \Delta, \epsilon)$ is a coalgebra such that $\Delta, \epsilon$ are algebra maps (or equivalently $\cdot, \eta$ are coalgebra maps).

**Definition 2.1.3.** A bialgebra $H$ is called a *Hopf algebra* if it is equipped with an 'antipode' map $S : H \to H$ such that $(Sh_{(1)})h_{(2)} = \epsilon(h) = h_{(1)}(Sh_{(2)})$ for all $h \in H$.

Let $H$ be a Hopf algebra with antipode $S$. It is known, e.g. in [19], that $S$ is unique and an anti-algebra map i.e. $S(ab) = (Sb)(Sa)$ for all $a, b \in H$. One also has $\Delta(Sa) = Sa_{(2)} \otimes Sa_{(1)}$, and $\epsilon S = \epsilon$. Furthermore, if $H$ is commutative or cocommutative, then $S^2 = \mathrm{id}$. Note that $S$ is not always invertible, but if $S^{-1}$ exists, then it become the antipode of Hopf algebra $H^{\mathrm{op}}$ or $H^{\mathrm{cop}}$. Moreover, Hopf algebra $H^{\mathrm{op/cop}}$ with opposite product and coopposite coproduct has the antipode $S$. Finally, a map $f : H \to A$ of two Hopf algebras $H$ and $A$ is called a Hopf algebra map if $f$ is both an algebra map and a coalgebra map such that $f \circ S_H = S_A \circ f$.

We will need a notion of duality between two Hopf algebras as given by a duality pairing.

**Definition 2.1.4.** Two bialgebras $A, H$ are *dually paired* if there is a bilinear map

$\langle\ ,\ \rangle : A \otimes H \to k$ called a *pairing* such that

$$\langle ab, h\rangle = \langle a \otimes b, \Delta h\rangle, \quad \langle a, hg\rangle = \langle \Delta a, h \otimes g\rangle, \quad \langle 1, h\rangle = \epsilon h, \quad \langle a, 1\rangle = \epsilon a,$$

for all $a, b \in A$ and $h \in H$. If $A, H$ are Hopf algebras, then additionally

$$\langle Sa, h\rangle = \langle a, Sh\rangle.$$

The above definition says that the algebra structure on $A$ corresponds to coalgebra structure of $H$ and vice versa. If $H$ is finite-dimensional with dual $H^* = \mathrm{Hom}(H, k)$, then the pairing $\langle\ ,\ \rangle : H^* \otimes H \to k$ becomes the evaluation map. Furthermore, $H^*$ becomes a bialgebra or Hopf algebra with product $\Delta^* : (H \otimes H)^* = H^* \otimes H^* \to H^*$ and coproduct $\cdot^* : H^* \to (H \otimes H)^* = H^* \otimes H^*$.

### 2.1.2 (co)modules over Hopf algebras

Let $H$ be a bialgebra or Hopf algebra. We first recall the following definition of modules which can be thought of as a polarisation of the definition of an algebra

**Definition 2.1.5.** By a left $H$-module $V$ we mean a vector space $V$ on which $H$ acts from the left by a left action $\triangleright : H \otimes V \to V$ satisfying $(hg)\triangleright v = h\triangleright(g\triangleright v)$, $1\triangleright v = v$.

Similarly for a right $H$-module $V$ with right action $\triangleleft : V \otimes H \to V$ satisfying $v\triangleleft(hg) = (v\triangleleft h)\triangleleft g$ and $v\triangleleft 1 = v$. We say that $V$ is an $H-H$-bimodule if it is both an $H$-left module and $H$-right module and these actions commute. Note that if $V$ is a left $H$-module and $H$ a Hopf algebra, then we can turn $V$ into a right $H$-module by $v\triangleleft h := (Sh)\triangleright v$. Also, if $V$ is a left $H$-module, then its dual $V^*$ is a right $H$-module by $(f\triangleleft h)(v) = f(h\triangleright v)$ for all $v \in V$, $f \in V^*$, and $h \in H$. Furthermore, if $H$ is a bialgebra or Hopf algebra, we can take the tensor product of two left $H$-modules $V$ and $W$ so that $V \otimes W$ is a $H$-modules with action $h\triangleright(v \otimes w) = h_{(1)}\triangleright v \otimes h_{(2)}\triangleright w$. We can let $V$ to have an algebra or coalgebra structure such that the product or coproduct of $V$ is compatible with the action of $H$.

**Definition 2.1.6.** An algebra $V$ is called a *left $H$-module algebra* if it is a left $H$-module, and additionally

$$h{\triangleright}(vw) = (h_{(1)}{\triangleright}v)(h_{(2)}{\triangleright}w), \quad h{\triangleright}1 = (\epsilon h)1 \tag{2.1.1}$$

for all $v, w \in V$ and $h \in H$. Similarly, an algebra $V$ is called a right $H$-right module algebra if it is a right $H$-module, and additionally

$$(vw){\triangleleft}h = (v{\triangleleft}h_{(1)})(w{\triangleleft}h_{(2)}), \quad 1{\triangleleft}h = 1(\epsilon h). \tag{2.1.2}$$

**Definition 2.1.7.** A coalgebra $V$ is called a *left $H$-module coalgebra* if it is a left $H$-module, and additionally

$$\Delta(h{\triangleright}v) = (h_{(1)}{\triangleright}v_{(1)}) \otimes (h_{(2)}{\triangleright}v_{(2)}), \quad \epsilon(h{\triangleright}v) = (\epsilon h)(\epsilon v) \tag{2.1.3}$$

for all $v \in V$ and $h \in H$. Similarly, a coalgebra $V$ is a right $H$-module coalgebra if it is a right $H$-module and additionally

$$\Delta(v{\triangleleft}h) = (v_{(1)}{\triangleleft}h_{(1)}) \otimes (v_{(2)}{\triangleleft}h_{(2)}), \quad \epsilon(v{\triangleleft}h) = (\epsilon v)(\epsilon h). \tag{2.1.4}$$

One can now reverse the arrow in the axioms for modules, landing us in the dual notion of comodules over a bialgebra or Hopf algebra $H$.

**Definition 2.1.8.** A right $H$-comodule is a vector space $V$ equipped with a right coaction $\Delta_R : V \to V \otimes H$, denoted by $\Delta_R v = v^{\overline{(0)}} \otimes v^{\overline{(1)}}$ such that $(\Delta_R \otimes \mathrm{id})\Delta_R = (\mathrm{id} \otimes \Delta)\Delta_R$ and $(\mathrm{id} \otimes \epsilon)\Delta_R = \mathrm{id}$, or equivalently

$$v^{\overline{(0)}\,\overline{(0)}} \otimes v^{\overline{(0)}\,\overline{(1)}} \otimes v^{\overline{(1)}} = v^{\overline{(0)}} \otimes v^{\overline{(1)}}{}_{(1)} \otimes v^{\overline{(1)}}{}_{(2)}, \quad v^{\overline{(0)}}\epsilon(v^{\overline{(1)}}) = v.$$

This is just a polarisation of the coalgebra axioms. Similarly, $V$ is a left $H$-comodule if it is equipped with left coaction $\Delta_L : V \to H \otimes V$ denoted by $\Delta_L v = v^{\overline{(1)}} \otimes v^{\overline{(\infty)}}$, such that $(\mathrm{id}\otimes\Delta_L)\Delta_L = (\Delta\otimes\mathrm{id})\Delta_L$ and $(\epsilon\otimes\mathrm{id})\Delta_L = \mathrm{id}$ or equivalently

$$v^{\overline{(1)}} \otimes v^{\overline{(\infty)}\overline{(1)}} \otimes v^{\overline{(\infty)}\overline{(\infty)}} = v^{\overline{(1)}}{}_{(1)} \otimes v^{\overline{(1)}}{}_{(2)} \otimes v^{\overline{(\infty)}}, \quad \epsilon(v^{\overline{(1)}})v^{\overline{(\infty)}} = v.$$

We say that $V$ is an $H-H$ bicomodule if it is both a left and right $H$-comodule such that these coactions commute. Note that in the Hopf algebra case, we can turn a right $H$-comodule $V$ into left $H$-comodule by $\Delta_L v := Sv^{\overline{(1)}} \otimes v^{\overline{(0)}}$. We can also take the tensor product of two right $H$-comodules $V, W$ so that $V\otimes W$ is a right $H$-module with coaction $\Delta_R(v\otimes w) = v^{\overline{(0)}} \otimes w^{\overline{(0)}} \otimes v^{\overline{(1)}}w^{\overline{(1)}}$. Similar to the module case, we can let $V$ to be an algebra or coalgebra such that its product or coproduct are compatible with coaction

**Definition 2.1.9.** An algebra $V$ is a *right $H$-comodule algebra* if $V$ is a right $H$-comodule and $\Delta_R$ is an algebra map, i.e.,

$$\Delta_R(vw) = v^{\overline{(0)}}w^{\overline{(0)}} \otimes v^{\overline{(1)}}w^{\overline{(1)}}, \quad \Delta_R(1) = 1\otimes 1, \tag{2.1.5}$$

for all $v, w \in V$. Similarly, an algebra $V$ is a left $H$-comodule algebra if $V$ is a left comodule, with $\Delta_L$ is an algebra map, i.e.,

$$\Delta_L(vw) = v^{\overline{(1)}}w^{\overline{(1)}} \otimes v^{\overline{(\infty)}}w^{\overline{(\infty)}}, \quad \Delta_L(1) = 1\otimes 1, \tag{2.1.6}$$

for all $v, w \in A$.

**Definition 2.1.10.** A coalgebra $V$ is said to be a *right $H$-comodule coalgebra* if $V$ is a right $H$-comodule with $\Delta_R$ such that

$$(\Delta\otimes\mathrm{id})\Delta_R(v) = v_{(1)}{}^{\overline{(0)}} \otimes v_{(2)}{}^{\overline{(0)}} \otimes v_{(1)}{}^{\overline{(1)}}v_{(2)}{}^{\overline{(1)}}, \quad (\epsilon\otimes\mathrm{id})\Delta_R = \epsilon\otimes 1 \tag{2.1.7}$$

for all $v \in V$. Similarly, a coalgebra $V$ is a left $H$-comodule if $V$ is a left $H$-comodule with $\Delta_L$ such that

$$(\mathrm{id}\otimes\Delta)\Delta_L(v) = v_{(1)}{}^{\overline{(1)}}v_{(2)}{}^{\overline{(1)}} \otimes v_{(1)}{}^{\overline{(\infty)}} \otimes v_{(2)}{}^{\overline{(\infty)}}, \quad (\mathrm{id}\otimes\epsilon)\Delta_L = 1\otimes\epsilon. \tag{2.1.8}$$

If $H$ is dually paired with $A$, then a right $H$-comodule $V$ becomes a left $A$-module by $a \triangleright v = \langle a, v^{\overline{(1)}} \rangle v^{\overline{(0)}}$. If $V$ is a right $H$-comodule algebra, then it becomes a left $A$-module algebra. Furthermore, if $V$ is finite-dimensional, then $V^*$ is a right $A$-module coalgebra. Similarly if $V$ is a right $H$-comodule coalgebra, then it becomes a left $A$-module coalgebra. Furthermore, $V^*$ is a right $A$-module algebra. There is also a notion of Hopf $H$-modules for Hopf algebra $H$, which requires compatibility between their modules and comodules structure.

**Definition 2.1.11.** Let $H$ be a Hopf algebra. A vector space $V$ is called a *right Hopf $H$-module* if it is both a right $H$-module and right $H$-comodule such that its coaction is a right $H$-module map, i.e. $\Delta_R(v \triangleleft h) = (\Delta_R v)(\Delta h)$. There is also an equivalent definition for a left Hopf $H$-module.

**Lemma 2.1.12** (Hopf Module Lemma)**.** *Let $V$ be a right Hopf $H$-module, and let*

$$V^{\mathrm{co}H} = \{v \in V | \Delta_R v = v \otimes 1\}$$

*be a space of right-invariant of $V$. Then $V \cong V^{\mathrm{co}H} \otimes H$ as right Hopf $H$-module, where*

$$(v \otimes h) \triangleleft g = v \otimes hg, \quad \Delta_R(v \otimes h) = v \otimes h_{(1)} \otimes h_{(2)}$$

*for all $v \in V$ and $h, g \in H$.*

*Proof.* This is well-known, for instance see [1]. The isomorphism $V^{\mathrm{co}H} \otimes H \to V$ is given by $v \otimes h \mapsto v \triangleleft h$, with inverse $v \mapsto v^{\overline{(0)}\overline{(0)}} \triangleleft S v^{\overline{(0)}\overline{(1)}} \otimes v^{\overline{(1)}}$. □

We also need another concept of compatibility between modules and comodules, which is called a *crossed module* or *Radford-Drinfeld-Yetter module*(c.f. [19, 25, 42]).

**Definition 2.1.13.** Let $H$ be a Hopf algebra. A vector space $V$ is called a right $H$-crossed module if $V$ is a right $H$-module and right $H$-coaction such that

$$\Delta_R(v \triangleleft h) = v^{\overline{(0)}} \triangleleft h_{(2)} \otimes (S h_{(1)}) v^{\overline{(1)}} h_{(3)},$$

for all $v \in V$ and $h \in H$. In this case, there is a morphism $\Psi : V \otimes V \to V \otimes V$ called a pre-braiding of $V$ given by

$$\Psi(v \otimes w) = w^{\overline{(0)}} \otimes v \triangleleft w^{\overline{(1)}}.$$

Similarly, $V$ is called a left $H$-crossed module if $V$ is a left $H$-module and left $H$-comodule such that

$$\Delta_L(h \triangleright v) = h_{(1)} v^{\overline{(1)}} Sh_{(3)} \otimes h_{(2)} \triangleright v^{\overline{(\infty)}},$$

with pre-braiding given by

$$\Psi(v \otimes w) = v^{\overline{(1)}} \triangleright w \otimes v^{\overline{(\infty)}}.$$

We will see later that the category of right $H$-crossed modules is a pre-braided category with the above pre-braidings, and becomes a braided category if the antipode of $H$ is invertible with the inverse braiding given by $\Psi^{-1}(v \otimes w) = w \triangleleft S^{-1} v^{\overline{(1)}} \otimes v^{\overline{(0)}}$. Similarly for left crossed modules.

**Example 2.1.14.** Every Hopf algebra $H$ can be made into a right $H$-crossed module with the right action given by multiplication, i.e. $h \triangleleft g = hg$ and the right coaction given by the adjoint coaction i.e. $\Delta_R = \mathrm{Ad}_R : H \to H \otimes H$ is given by $\mathrm{Ad}_R(h) = h_{(2)} \otimes (Sh_{(1)}) h_{(3)}$. Furhermore, we can restrict $H$ to its *augmentation ideal* $H^+ = \ker \epsilon$, making $H^+$ a right $H$-crossed module.

In later chapters we will need a notion of super-Hopf algebra, which is a $\mathbb{Z}_2$-graded Hopf algebra $H$, where $H = H_0 \oplus H_1$ with grade $|a| = i$ for $a \in H_i$, where $i = 0, 1$. Here the coproduct respects the total grades and is an algebra map to the super tensor product algebra

$$(h \otimes g)(h' \otimes g') = (-1)^{|g||h'|}(hh' \otimes gg')$$

for $h, h', g, g' \in H$. The counit also respects the grade and hence $\epsilon|_{H_1} = 0$ if we work over a field $k$. If $H$ is a super-Hopf algebra then $H\underline{\otimes}H$ is also a super-Hopf algebra, with

$$\Delta(h \otimes g) = (-1)^{|h_{(2)}||g_{(1)}|} h_{(1)} \otimes g_{(1)} \otimes h_{(2)} \otimes g_{(2)}.$$

The notions of action and coaction under a Hopf algebra similarly have super versions, namely respecting the total degree. All of the above further notions similarly have super versions with transposition acquring an extra sign depending on degrees. For instance in the super version of right $H$-crossed module $V$, the compatibility between action and coaction is given by

$$\Delta_R(v \triangleleft h) = (-1)^{|v^{\overline{(1)}}|(|h_{(1)}|+|h_{(2)}|)+|h_{(1)}||h_{(2)}|} v^{\overline{(0)}} \triangleleft h_{(2)} \otimes (Sh_{(1)}) v^{\overline{(1)}} h_{(3)}$$

for all homogeneous $v \in V$ and $h \in H$.

## 2.2 (Co)quasitriangular Hopf algebra

We first recall Drinfeld's definition of an important class of noncocommutative Hopf algebras.

**Definition 2.2.1.** A Hopf algebra $H$ is called *quasitriangular* [11] if equipped with invertible $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ (summation understood) such that

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.2.1}$$

$$\mathrm{flip} \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}. \tag{2.2.2}$$

Here the leg indices indicate where the factor of $\mathcal{R}$ lives in the tensor product, for instance $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{13} = \mathcal{R}^{(1)} \otimes 1 \otimes \mathcal{R}^{(2)}$, etc.

Basic properties of quasitriangular Hopf algebras are well-studied, for example in [20]. We have $(\epsilon \otimes \mathrm{id})\mathcal{R} = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{R}$. We also have $(S \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1}$ and $(\mathrm{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R}$ which directly imply $(S \otimes S)\mathcal{R} = \mathcal{R}$. Furthermore, any quasitriangular Hopf algebra

$(H, \mathcal{R})$ satisfies the following quantum Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.2.3}$$

It can be shown that the antipode of $H$ is invertible [19, 24]. Moreover $H^{\mathrm{op}}$ and $H^{\mathrm{cop}}$ are also quasitriangular with quasitriangularity $\mathcal{R}^{-1}$. We denote by $\overline{H}$ the quasitriangular Hopf algebra which is the same Hopf algebra as $H$ but with quasitriangular structure $\overline{\mathcal{R}} = \mathcal{R}_{21}^{-1}$. We also need the notion of coquasitriangular Hopf algebra :

**Definition 2.2.2.** A Hopf algebra $A$ is called *coquasitriangular* if it is equipped with a convolution-invertible map $\mathcal{R} : A \otimes A \to k$ satisfying

$$\mathcal{R}(ab, c) = \mathcal{R}(a, c_{(1)})\mathcal{R}(b, c_{(2)}), \quad \mathcal{R}(a, bc) = \mathcal{R}(a_{(1)}, c)\mathcal{R}(a_{(2)}, b), \tag{2.2.4}$$

$$a_{(1)}b_{(1)}\mathcal{R}(b_{(2)}, a_{(2)}) = \mathcal{R}(b_{(1)}, a_{(1)})b_{(2)}a_{(2)} \tag{2.2.5}$$

for all $a, b, c \in A$.

One can see that $\mathcal{R}(a, 1) = \epsilon a = \mathcal{R}(1, a)$. Also, $\mathcal{R}(a, Sb) = \mathcal{R}^{-1}(a, b)$, $\mathcal{R}^{-1}(a, Sb) = \mathcal{R}(a, b)$ and $\mathcal{R}(Sa, Sb) = \mathcal{R}(a, b)$. It also obeys the following dual quantum Yang-Baxter equation

$$\mathcal{R}(a_{(1)}, b_{(1)})\mathcal{R}(a_{(2)}, c_{(1)})\mathcal{R}(b_{(2)}, c_{(2)}) = \mathcal{R}(b_{(1)}, c_{(1)})\mathcal{R}(a_{(1)}, c_{(2)})\mathcal{R}(a_{(2)}, b_{(2)}). \tag{2.2.6}$$

Note that the antipode of $A$ is invertible, and $A^{\mathrm{cop}}$ and $A^{\mathrm{op}}$ are coquasitriangular Hopf algebras with coquasitriangular structure $\mathcal{R}^{-1}$. We also denote $\overline{A}$ to be the same Hopf algebra as $A$ but with coquasitriangular structure $\overline{\mathcal{R}} = \mathcal{R}_{21}^{-1}$ i.e. $\overline{\mathcal{R}}(a, b) = \mathcal{R}(Sb, a)$ for all $a, b \in \overline{A}$.

**Lemma 2.2.3.** *If $H$ is a finite-dimensional quasitriangular Hopf algebra, then $H^*$ is a coquasitriangular Hopf algebra. Here $\mathcal{R}(a, b) = \langle \mathcal{R}^{(1)}, a\rangle\langle \mathcal{R}^{(2)}, b\rangle$.*

## 2.3 Braided Hopf algebras

We refer to [18] for a full treatment of category theory. Recall that a category $\mathcal{C}$ consists of the following data : (i) a class of objects (ii) a set of morphism $\mathcal{C}(X,Y)$ for any two objects $X,Y \in \mathcal{C}$ (iii) for any three objects $X,Y,Z \in \mathcal{C}$, there is a map $\mathcal{C}(X,Y)\times\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ by $(f,g)\mapsto g\circ f$ such that $(h\circ g)\circ f = h\circ(g\circ f)$ for all $f : X \to Y$ in $\mathcal{C}(X,Y)$ and $g : Y \to Z$ in $\mathcal{C}(Y,Z)$, and for all $X \in \mathcal{C}$ there exists $\mathrm{id}_X \in \mathcal{C}(X,X)$ such that $f\circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f \in \mathcal{C}(X,Y)$.

A (covariant) *functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two categories assigns to any object $X \in \mathcal{C}$ another object $F(X) \in \mathcal{D}$, and assigns to any morphism $f \in \mathcal{C}(X,Y)$ another morphism $F(f) \in \mathcal{D}(F(X),F(Y))$ such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g\circ f) = F(g)\circ F(f)$.

A *natural transformation* $\theta : F \Rightarrow G$ between two functors $F,G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\{\theta_X : F(X) \to G(X) | X \in \mathcal{C}\}$ in $\mathcal{D}$ such that $\theta_Y \circ F(f) = G(f)\circ\theta_X$ for any morphism $f \in \mathcal{C}(X,Y)$. The natural transformation $\theta$ is called a *natural isomorphism* if each $\theta_X$ is invertible as a morphism.

**Definition 2.3.1.** A *monoidal* category $(\mathcal{C},\otimes,\underline{1},\Phi,l,r)$ is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a natural transformation (called an *associator*), $\Phi : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$ i.e. a collection of functorial isomorphisms

$$\Phi_{X,Y,Z} : (X\otimes Y)\otimes Z \cong X\otimes(Y\otimes Z),$$

and unit object $\underline{1}$ with its associated natural isomorphisms $l : \mathrm{id} \Rightarrow -\otimes\underline{1}$ and $r : \mathrm{id} \Rightarrow \underline{1}\otimes-$ such that the Pentagon and Triangle identities shown in Figure 2.1 and Figure 2.2 are satisfied.

**Definition 2.3.2.** A *braided category* $(\mathcal{C},\Psi)$ is a monoidal category $\mathcal{C}$ with a natural isomorphism $\Psi : \otimes \Rightarrow \otimes^{\mathrm{op}}$, i.e. a collection of functorial isomorphisms $\Psi_{X,Y} : X\otimes Y \cong Y\otimes X$ for all $X,Y \in \mathcal{C}$ such that the Hexagon identities shown in Figure 2.3 are satisfied. Here the inverse braiding means $\Psi^{-1}_{Y,X} : Y\otimes X \to X\otimes Y$ such that $\Psi^{-1}_{Y,X}\circ\Psi_{X,Y} = \mathrm{id}_{X\otimes Y}$ and $\Psi_{X,Y}\circ\Psi^{-1}_{Y,X} = \mathrm{id}_{Y,X}$. If we do not assume $\Psi$ to be invertible, then we say the

$$\begin{tikzcd}
& (X\otimes Y)\otimes(Z\otimes U) \arrow[dr, "\Phi"] & \\
((X\otimes Y)\otimes Z)\otimes U \arrow[ur, "\Phi"] \arrow[d, "\Phi\otimes\mathrm{id}"] & & X\otimes(Y\otimes(Z\otimes U)) \\
(X\otimes(Y\otimes Z))\otimes U \arrow[rr, "\Phi"] & & X\otimes((Y\otimes Z)\otimes U) \arrow[u, "\mathrm{id}\otimes\Phi"']
\end{tikzcd}$$

Figure 2.1: Pentagon identity for monoidal category

$$\begin{tikzcd}
(X\otimes\underline{1})\otimes Y \arrow[rr, "\Phi"] & & X\otimes(\underline{1}\otimes Y) \\
& X\otimes Y \arrow[ul, "l\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes r"'] &
\end{tikzcd}$$

Figure 2.2: Triangle identity for monoidal category

$$\begin{tikzcd}
& X\otimes(Y\otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
X\otimes(Z\otimes Y) \arrow[d, "\Phi^{-1}"'] & & (X\otimes Y)\otimes Z \arrow[d, "\Psi"] \\
(X\otimes Z)\otimes Y \arrow[dr, "\Psi\otimes\mathrm{id}"'] & & Z\otimes(X\otimes Y) \arrow[dl, "\Phi^{-1}"] \\
& (Z\otimes X)\otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (X\otimes Y)\otimes Z \arrow[dl, "\Phi"'] \arrow[dr, "\Psi\otimes\mathrm{id}"] & \\
X\otimes(Y\otimes Z) \arrow[d, "\Psi"] & & (Y\otimes X)\otimes Z \arrow[d, "\Phi"'] \\
(Y\otimes Z)\otimes X \arrow[dr, "\Phi"'] & & Y\otimes(X\otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"] \\
& Y\otimes(Z\otimes X) &
\end{tikzcd}$$

Figure 2.3: Hexagon identities for braided category

category $\mathcal{C}$ is *pre-braided*.

One can consider objects in $\mathcal{C}$ to enjoy some algebraic structure in the sense that the morphisms in $\mathcal{C}$ satisfy the axioms of the said algebraic structure. Thus, an algebra in a braided category $\mathcal{C}$ is an object $B$ with a product morphism $\cdot : B\otimes B\to B$ satisfying the axiom for associativity, and unit morphism $\eta:\underline{1}\to B$ satisfying unity axiom. For any algebra objects $B, C$ in $\mathcal{C}$, their braided tensor product $B\underline{\otimes}C$ is also an algebra in $\mathcal{C}$

with product morphism $\cdot$ such that [19, 24]

$$\cdot_{B\underline{\otimes}C} : (\cdot_B \otimes \cdot_C) \circ (\mathrm{id}_B \otimes \Psi_{C,B} \otimes \mathrm{id}_C)$$

and tensor product unit morphism. This is a generalisation of the super tensor product algebra where the transposition by degree is replaced by more general braiding.

Similarly, a coalgebra in a braided category $\mathcal{C}$ is an object $B$ with a coproduct morphism $\underline{\Delta} :\to B \otimes B$ and counit morphism $\underline{\epsilon} : C \to \underline{1}$ such that they satisfy coassociativity and unity axioms. For any coalgebra objects $B, C$ in $\mathcal{C}$, $B\underline{\otimes}C$ is also a coalgebra in $\mathcal{C}$ with coproduct morphism

$$\underline{\Delta}_{B\underline{\otimes}C} = (\mathrm{id}_B \otimes \Psi_{B,C} \otimes \mathrm{id}) \circ (\underline{\Delta}_B \otimes \underline{\Delta}_C)$$

and tensor product counit morphism, which generalises the super tensor product coalgebra. Now we are ready to define braided Hopf algebras :

**Definition 2.3.3.** [19, 24, 31] $(B, \cdot, \eta, \underline{\Delta}, \underline{\epsilon}, \underline{S})$ in a braided monoidal category $\mathcal{C}$ is called a *braided Hopf algebra* if $(B, m, \eta)$ is an algebra in $\mathcal{C}$, $(B, \underline{\Delta}, \underline{\epsilon})$ is a coalgebra in $\mathcal{C}$, $\underline{\Delta}$ is an algebra morphism to the braided tensor product algebra, and the antipode morphism $\underline{S} : B \to B$ obeying $\cdot(\mathrm{id} \otimes S)\underline{\Delta} = \mathrm{id} = \cdot(S \otimes \mathrm{id})\underline{\Delta}$.

In a concrete setting we write $\underline{\Delta}b = b_{\underline{(1)}} \otimes b_{\underline{(2)}}$ (summation understood), so that for example the $\underline{\Delta}(bc) = b_{\underline{(1)}}\Psi(b_{\underline{(2)}} \otimes c_{\underline{(1)}})c_{\underline{(2)}}$ is the bialgebra axiom for braided Hopf algebra, with $\Psi$ the braiding on $B \otimes B$. There is a diagrammatic method for braided Hopf algebras which we will not use explicitly. It can be used, for example, to proved the following basic identities [24, 31],

$$\underline{S} \circ \cdot = \cdot \circ \Psi \circ (\underline{S} \otimes \underline{S}), \quad \underline{\Delta} \circ \underline{S} = (\underline{S} \otimes \underline{S}) \circ \Psi \circ \underline{\Delta}. \tag{2.3.1}$$

If $\mathcal{C}$ is the category of super or $\mathbb{Z}_2$-graded vector spaces, the above definition gives us a super-Hopf algebra since the categorical braiding on $B$ is $\Psi(b \otimes c) = (-1)^{|b||c|} c \otimes b$. We will also need the notion of super braided Hopf algebra where the object satisfies the same axioms as braided Hopf algebra but $\mathbb{Z}_2$-graded and with an extra factor $(-1)^{|\ ||\ |}$ in every braiding. This assumes direct sums in the category and can be viewed as a braided Hopf algebra in a super version of the braided category where objects are of the form $B = B_0 \oplus B_1$ and the braiding has additional signs according to the degree.

## 2.4 Bosonisation

We first recall that if $B$ is a right $H$-module algebra, there is a right cross product algebra $H \triangleright\!\!< B$ built on $H \otimes B$ with product

$$(h \otimes b)(g \otimes c) = hg_{(1)} \otimes (b \triangleleft g_{(2)})c \tag{2.4.1}$$

for all $h, g \in H$ and $b, c \in B$. There is an equivalent left cross product $B >\!\!\triangleleft H$ where $B \in {}_H\mathcal{M}$ as module algebra with product is given by

$$(b \otimes h)(c \otimes g) = b(h_{(1)} \triangleright c) \otimes h_{(2)}g \tag{2.4.2}$$

Similarly, if $B$ is a right $H$-comodule coalgebra, there is a right cross coproduct coalgebra $H \blacktriangleright\!\!< B$ with coproduct

$$\Delta(h \otimes b) = h_{(1)} \otimes b_{(1)}{}^{\overline{(0)}} \otimes h_{(2)} b_{(1)}{}^{\overline{(1)}} \otimes b_{(2)}. \tag{2.4.3}$$

And if $B$ is a left $H$-comodule coalgebra, there is a left cross coproduct coalgebra $B >\!\!\blacktriangleleft H$ with coproduct

$$\Delta(b \otimes h) = b_{(1)} \otimes b_{(2)}{}^{\overline{(1)}} h_{(1)} \otimes b_{(2)}{}^{\overline{(\infty)}} \otimes h_{(2)}. \tag{2.4.4}$$

**Lemma 2.4.1** (Bosonisation)**.** *[25, 26] Let $H$ be an ordinary Hopf algebra and $B$ be a braided Hopf algebra in $\mathcal{M}_H^H$ the category of right $H$-crossed modules, then there is an ordinary Hopf algebra $H \triangleright\!\!\!\blacktriangleleft B$, the* biproduct *or* bosonisation *of $B$ built on $H \otimes B$ with product the cross product $H \triangleright\!\!< B$ and coproduct the cross coproduct $H \blacktriangleright\!\!< B$ by the assumed* *action and coaction of $H$. Moreover, there is a Hopf algebra projection $H \overset{\hookrightarrow}{\twoheadleftarrow} H{\triangleright\!\!\!<}B$.*

This structure was first found by Radford in [42] from a study of Hopf algebras with projection (they are of this form) prior to the theory of braided Hopf algebras, while the above formulation is due to Majid in [25]. There is also a left-handed version of biproduct $B{>\!\!\!\triangleleft\kern-.33em\cdot} H$ with product is cross product $B{>\!\!\!\triangleleft} H$ and coproduct is cross coproduct $B{>\!\!\blacktriangleleft} H$, where $B$ is braided Hopf algebra in category of left $H$-crossed modules ${}^H_H\mathcal{M}$.

In the case of $H$ (resp. $A$) being a quasitriangular (resp. coquasitriangular) Hopf algebra, their category of left/right (co)modoules is a braided monoidal category with the following braidings

$$\Psi_L(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v, \qquad \Psi_R(v \otimes w) = w \triangleleft \mathcal{R}^{(1)} \otimes v \triangleleft \mathcal{R}^{(2)},$$
$$\Psi^L(v \otimes w) = \mathcal{R}(w^{\overline{(1)}}, v^{\overline{(1)}}) w^{\overline{(\infty)}} \otimes v^{\overline{(\infty)}}, \quad \Psi^R(v \otimes w) = w^{\overline{(0)}} \otimes v^{\overline{(0)}} \mathcal{R}(v^{\overline{(1)}}, w^{\overline{(1)}})$$

where $\Psi_L$ is the braiding for the left $H$-modules category ${}_H\mathcal{M}$, similarly $\Psi_R$ for right $H$-modules category $\mathcal{M}_H$, and $\Psi^L$ for the left $A$-comodule category ${}^A\mathcal{M}$, similarly $\Psi^R$ for right-comodules $\mathcal{M}^A$. There is a braided monoidal functor ${}_H\mathcal{M} \hookrightarrow {}^H_H\mathcal{M}$ in [19, 27] with a coaction induced by the quasitriangular structure of $H$ so as to form a crossed module. Similarly from the right. Explicitly, these induced coactions are given by

$$\Delta_L b = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b, \quad \Delta_R b = b \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \tag{2.4.5}$$

for all $b \in B$. Thus for braided Hopf algebra $B \in \mathcal{M}_H \hookrightarrow \mathcal{M}^H_H$, the right bosonisation $H{\triangleright\!\!\!<}B$ has cross product $H{\triangleright\!\!\!<}B$ by the given action, and a cross coproduct $H{\blacktriangleright\!\!<}B$ by the induced right coaction to give

$$\Delta(h \otimes b) = h_{(1)} \otimes b_{\underline{(1)}} \triangleleft \mathcal{R}^{(1)} \otimes h_{(2)} \mathcal{R}^{(2)} \otimes b_{\underline{(2)}}. \tag{2.4.6}$$

Similarly for a braided Hopf algebra $B \in {}_H\mathcal{M} \hookrightarrow {}^H_H\mathcal{M}$ to give a left bosonisation $B{>\!\!\!\triangleleft\kern-.33em\cdot} H$ with cross product $B{>\!\!\!\triangleleft} H$ by the given action, and cross coproduct $B{>\!\!\blacktriangleleft} H$ by the induced left coaction to give

$$\Delta(b\otimes h)=b_{\underline{(1)}}\otimes\mathcal{R}^{(2)}h_{(1)}\otimes\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}}\otimes h_{(2)}. \tag{2.4.7}$$

The same is true for $A$ being coquasitriangular Hopf algebra via functors ${}^{A}\mathcal{M}\hookrightarrow{}^{A}_{A}\mathcal{M}$ and $\mathcal{M}^{A}\hookrightarrow\mathcal{M}^{A}_{A}$, with the following induced actions

$$a\triangleright b=\mathcal{R}(b^{\overline{(1)}},a)b^{\overline{(\infty)}},\quad b\triangleleft a=b^{\overline{(0)}}\mathcal{R}(b^{\overline{(1)}},a) \tag{2.4.8}$$

for all $a\in A$ and $b\in B$. Thus for a braided Hopf algebra $B\in\mathcal{M}^{A}$, we have a *cobosonisation* $A\triangleright\!\!\!\blacktriangleleft B$ with $A\blacktriangleright\!\!<B$ by the given coaction and $A\triangleright\!\!<B$ by the induced action from the above functor. Explicitly, the cross product is

$$(a\otimes b)(d\otimes c)=ad_{(1)}\otimes b^{\overline{(0)}}c\mathcal{R}(b^{\overline{(1)}},d_{(2)}) \tag{2.4.9}$$

for all $a,d\in A$ and $b,c\in B$. Similarly for a braided Hopf algebra $B\in{}^{A}\mathcal{M}$, there is a cobosonisation $B{>\!\!\!\triangleleft} A$ with cross coproduct $B{>\!\!\blacktriangleleft}A$ by the given coaction, and cross product $B{>\!\!\!\triangleleft}A$ by the induced action, explicitly

$$(b\otimes a)(c\otimes d)=bc^{\overline{(\infty)}}\otimes a_{(2)}d\mathcal{R}(c^{\overline{(1)}},a_{(1)}). \tag{2.4.10}$$

Finally, the notion of a dually paired or categorical dual braided Hopf algebra $B^{\star}$ (when $B$ is a rigid object, e.g. finite-dimensional in our applications) in [24, 31] needs a little care to define the pairing $B^{\star}\otimes B^{\star}\otimes B\otimes B$ by pairing $B^{\star}\otimes B$ in the middle first. Pairing maps go to the trivial object. In our context, where objects are built on vector spaces, it is useful to match ordinary Hopf algebra conventions by defining $B^{*}$ with the adjoint algebra and coalgebra structures in the usual way rather than the above categorical way, which, however, canonically lands $B^{*}$ in a different category from $B$:

**Lemma 2.4.2.** *Let $B$ be a finite-dimensional braided Hopf algebra in $\mathcal{M}^{A}$ then the ordinary dual $B^{*}$ is a braided Hopf algebra in ${}^{A}\mathcal{M}$. Similarly, if $B\in{}^{A}\mathcal{M}$ then the* *ordinary dual $B^*$ is a braided Hopf algebra in $\mathcal{M}^A$.*

*Proof.* We check carefully that our conventions match up in the way stated. A right coaction on $B$ is equivalent to a left coaction on $B^*$ with the two related by

$$\langle x, b^{\overline{(0)}}\rangle\langle b^{\overline{(1)}}, a\rangle = \langle x^{\overline{(\infty)}}, b\rangle\langle x^{\overline{(1)}}, a\rangle \tag{2.4.11}$$

for all $x \in B^*$, $b \in B$, and $a \in A$. Since the duality is the ordinary one, it is clear that $B^*$ is an algebra and coalgebra with structure maps morphisms. We need to prove the coproduct homomorphism property for $B^* \in {}^A\mathcal{M}$,

$$\begin{aligned}
\langle\underline{\Delta}(xy), b\otimes c\rangle =& \langle x\otimes y, \underline{\Delta}(bc)\rangle = \langle x\otimes y, b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}} \otimes b_{\underline{(2)}}{}^{\overline{(0)}}c_{\underline{(2)}}\rangle\mathcal{R}(b_{\underline{(2)}}{}^{\overline{(1)}}, c_{\underline{(1)}}{}^{\overline{(1)}})\\
=&\langle x, b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}}\rangle\langle y, b_{\underline{(2)}}{}^{\overline{(0)}}c_{\underline{(2)}}\rangle\langle b_{\underline{(2)}}{}^{\overline{(1)}}, \mathcal{R}^{(1)}\rangle\langle c_{\underline{(1)}}{}^{\overline{(1)}}, \mathcal{R}^{(2)}\rangle\\
=&\langle x_{\underline{(1)}}, b_{\underline{(1)}}\rangle\langle x_{\underline{(2)}}, c_{\underline{(1)}}{}^{\overline{(0)}}\rangle\langle y_{\underline{(1)}}, b_{\underline{(2)}}{}^{\overline{(0)}}\rangle\langle y_{\underline{(2)}}, c_{\underline{(2)}}\rangle\langle b_{\underline{(2)}}{}^{\overline{(1)}}, \mathcal{R}^{(1)}\rangle\langle c_{\underline{(1)}}{}^{\overline{(1)}}, \mathcal{R}^{(2)}\rangle\\
=&\langle x_{\underline{(1)}}, b_{\underline{(1)}}\rangle\langle y_{\underline{(1)}}{}^{\overline{(\infty)}}, b_{\underline{(2)}}\rangle\langle x_{\underline{(2)}}{}^{\overline{(\infty)}}, c_{\underline{(1)}}\rangle\langle y_{\underline{(2)}}, c_{\underline{(2)}}\rangle\langle y_{\underline{(1)}}{}^{\overline{(1)}}, \mathcal{R}^{(1)}\rangle\langle x_{\underline{(2)}}{}^{\overline{(1)}}, \mathcal{R}^{(2)}\rangle\\
=&\langle x_{\underline{(1)}}y_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(\infty)}}y_{\underline{(2)}}, b\otimes c\rangle\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}).
\end{aligned}$$

for all $x, y \in B^*$, and for all $b, c \in B$. Similarly for the second part of the lemma. □

The same is true for $B \in \mathcal{M}_H$ to give $B^* \in {}_H\mathcal{M}$ which is part of the original theory in [28]. We will need the above dual version, so we have given that with proof. We then have a precise statement:

**Lemma 2.4.3.** *Let $H$ be finite-dimensional and quasitriangular with dual $A$, and $B$ be a finite-dimensional braided Hopf algebra in $\mathcal{M}_H$, then $(H{\triangleright\!\!\!<}B)^* = A{\triangleright\!\!\!<}B^*$. Similarly, if $B \in {}_H\mathcal{M}$ then $(B{>\!\!\!\triangleleft}H)^* = B^*{>\!\!\!\triangleleft}A$.*

*Proof.* We check that everything matches up correctly. The coproduct of $(H{\triangleright\!\!\!<}B)^*$ is defined by

$$\langle\Delta(k\otimes x), (h\otimes b)\otimes(g\otimes c)\rangle = \langle k\otimes x, (h\otimes b)(g\otimes c)\rangle$$

$$
\begin{aligned}
&= \langle k, hg_{(1)} \rangle \langle x, (b \triangleleft g_{(2)})c \rangle \\
&= \langle k_{(1)}, h \rangle \langle k_{(2)}, g_{(1)} \rangle \langle g_{(2)} \triangleright x_{\underline{(1)}}, b \rangle \langle x_{\underline{(2)}}, c \rangle \\
&= \langle k_{(1)}, h \rangle \langle k_{(2)}, g_{(1)} \rangle \langle x_{\underline{(1)}}{}^{\overline{(0)}}, b \rangle \langle x_{\underline{(1)}}{}^{\overline{(1)}}, g_{(2)} \rangle \langle x_{\underline{(2)}}, c \rangle \\
&= \langle k_{(1)} \otimes x_{\underline{(1)}}{}^{\overline{(0)}} \otimes k_{(2)} x_{\underline{(1)}}{}^{\overline{(1)}} \otimes x_{\underline{(2)}}, h \otimes b \otimes g \otimes c \rangle
\end{aligned}
$$

for all $b, c \in B$, $x, y \in B^*$, $h, g \in H$, and $k, \ell \in A$. Hence $\Delta(k \otimes x)$ agrees with the comultiplication of $A{\blacktriangleright\!\!<}B^*$. Similarly, the product of $(H{\blacktriangleright\!\!<}B)^*$ is defined by

$$
\begin{aligned}
\langle (k \otimes x)(\ell \otimes y), h \otimes b \rangle = \langle k \otimes x \otimes \ell \otimes y, \Delta(h \otimes b) \rangle \\
= \langle k, h_{(1)} \rangle \langle x, b_{\underline{(1)}} \triangleleft \mathcal{R}^{(1)} \rangle \langle \ell, h_{(2)} \mathcal{R}^{(2)} \rangle \langle y, b_{\underline{(2)}} \rangle \\
= \langle k, h_{(1)} \rangle \langle \mathcal{R}^{(1)} \triangleright x, b_{\underline{(1)}} \rangle \langle \ell_{(1)}, h_{(2)} \rangle \langle \ell_{(2)}, \mathcal{R}^{(2)} \rangle \langle y, b_{\underline{(2)}} \rangle \\
= \langle k, h_{(1)} \rangle \langle x^{\overline{(0)}}, b_{\underline{(1)}} \rangle \langle x^{\overline{(1)}}, \mathcal{R}^{(1)} \rangle \langle \ell_{(1)}, h_{(2)} \rangle \langle \ell_{(2)}, \mathcal{R}^{(2)} \rangle \langle y, b_{\underline{(2)}} \rangle \\
= \langle k\ell_{(1)} \otimes x^{\overline{(0)}} y, h \otimes b \rangle \mathcal{R}(x^{\overline{(1)}}, \ell_{(2)}).
\end{aligned}
$$

Hence $(k \otimes x)(\ell \otimes y)$ agrees with the multiplication in $A{\blacktriangleright\!\!<}B^*$. The proof of the left-handed version is similar. □

## 2.5 Differentials of Hopf algebras

**Definition 2.5.1.** A first order differential calculus on an algebra $A$ is an $A-A$-bimodule $\Omega^1$ equipped with a differential map $\mathrm{d} : A \to \Omega^1$ satisfying Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ such that $\Omega^1 = \mathrm{span}\{a\mathrm{d}b\}$.

This is the minimum data needed to generalise classical differentials. We also say that $\mathrm{d}a$ is a *1-form.* One can also easily find that $\mathrm{d}1 = 0$ by applying Leibniz rule to $1.1 = 1$. It is essential to not assume $\mathrm{d}b.a = a\mathrm{d}b$, since if otherwise, then we have a large kernel if our algebra is noncommutative since $\mathrm{d}(ab - ba) = 0$. For every algebra $A$, there is the universal calculus $\Omega^1_{\mathrm{uni}}$ given by $\Omega^1_{\mathrm{uni}} = \ker(\,\cdot\,) \subset A \otimes A$ with $\mathrm{d}_{\mathrm{uni}} a = 1 \otimes a - a \otimes 1$. It is universal since any other calculus of $A$ is isomorphic to $\Omega^1_{\mathrm{uni}}/I$ for some sub-bimodule $I$

of $\Omega^1_{\text{uni}}$.

**Definition 2.5.2.** A calculus $\Omega^1$ on an algebra $A$ is said to be inner if there is an element $\theta \in \Omega^1$ such that $\mathrm{d}a = [\theta, a] = \theta a - a\theta$ for all $a \in A$

In the case of $A$ being a Hopf algebra, its differential calculus $\Omega^1$ becomes right/left Hopf $A - A$ bimodule in the case that $\Omega^1$ is covariant.

**Definition 2.5.3.** A calculus $\Omega^1$ on a Hopf algebra $A$ is called *right-covariant* if $\Omega^1$ is a right $A$-comodule with right coaction $\Delta_R : \Omega^1 \to \Omega^1 \otimes A$ such that $\Delta_R$ commutes with d in the sense $\Delta_R \mathrm{d} = (\mathrm{d} \otimes \mathrm{id})\Delta$. Similarly for a left-covariant calculus. We call $\Omega^1$ *bicovariant* if it is both right and left covariant.

Let $\Lambda^1 = \{v \in \Omega^1 | \Delta_R v = v \otimes 1\}$ be the space of invariant 1-forms of $\Omega^1$. By the Hopf Module Lemma, any right covariant $\Omega^1$ is a free module over $\Lambda^1$ with $\mathrm{d}a = (\varpi \pi_\epsilon a_{(1)})a_{(2)}$, where $\pi_\epsilon = \mathrm{id} - \epsilon : A \to A^+$ and $\varpi : A^+ \twoheadrightarrow \Lambda^1$ is given by $\varpi(a) = (\mathrm{d}a_{(1)})Sa_{(2)}$, the *Maurer-Cartan form.* Moreover, any bicovariant $\Omega^1$ is an $A - A$- bimodule and bicomodule. Thus by the Hopf Module Lemma, there is an object $M$ in the category of crossed modules $\mathcal{M}^A_A$ such that $\Omega^1 \cong M \otimes A$. Note that from Example 2.1.14, $A^+$ is an $A$-crossed module, and due to Woronowicz [47], one can identify $M$ as quotient module $A^+$ of $A$. Thus, one has $\Omega^1 \cong A^+/I \otimes A$ for some ad-stable right ideal $I$.

### 2.5.1 Exterior Algebras

**Definition 2.5.4.** A *differential graded algebra (DGA)* over algebra $A$ is a graded algebra $\Omega = \oplus_{n \geq 0} \Omega^n$ equipped with $\mathrm{d} : \Omega^n \to \Omega^{n+1}$ such that $\mathrm{d}^2 = 0$ and the graded Leibniz rule $\mathrm{d}(\omega\eta) = (\mathrm{d}\omega)\eta + (-1)^{|\omega|}\omega \mathrm{d}\eta$ holds for all $\omega, \eta \in \Omega$. Here $|\omega|$ denotes the degree of $\omega$. Furthermore, we say $\Omega$ is an *exterior algebra* if $\Omega$ is generated by $A$ and $\mathrm{d}A$.

Given $(A, \Omega^1, \mathrm{d})$, there is a maximal prolongation exterior algebra $\Omega_{\max}$ where we impose the quadratic relation $\sum_i \mathrm{d}b_i \mathrm{d}c_i + \sum_j \mathrm{d}r_j \mathrm{d}s_j$ whenever $\sum_i \mathrm{d}b_i.c_i - \sum_j r_j \mathrm{d}s_j = 0$ is a relation in $\Omega^1$, where $b_i, c_i, r_i, s_i \in A$. We also have the left/right/bicovariant exterior algebras :

**Definition 2.5.5.** An exterior algebra $\Omega$ on $A$ is called left covariant if it is a left $A$-comodule algebra with graded $\Delta_L$ commuting with d, and similarly for the right covariant case. It is bicovariant if it is both left-covariant and right-covariant.

In the case $\Omega^1$ is a bicovariant calculus, it is known [8] that $\Omega$ is a super Hopf algebra with

$$\Delta_*|_A = \Delta, \quad \Delta_*|_{\Omega^1} = \Delta_L + \Delta_R.$$

The comultiplication on higher degrees are obtained from the coproduct on degree 0, 1, for example we have

$$\Delta_*(\mathrm{d}a\mathrm{d}b) = \Delta_*(\mathrm{d}a)\Delta_*(\mathrm{d}b).$$

We also need a concept of *strongly bicovariant* exterior algebra which is first introduced in [39]

**Definition 2.5.6.** An exterior algebra $\Omega$ is *strongly bicovariant* [39] if it is a super-Hopf algebra with super-degree given by the grade mod 2, super-coproduct $\Delta_*$ grade preserving and restricting to the coproduct of $A$, and if d is a super-coderivation in the sense

$$\Delta_*\mathrm{d}\omega = (\mathrm{d} \otimes \mathrm{id} + (-1)^{|\ |}\mathrm{id} \otimes \mathrm{d})\Delta_*\omega. \tag{2.5.1}$$

It is proved in [39] that any strongly bicovariant exterior algebra is bicovariant, justifying the terminology, with $\Delta_L, \Delta_R$ on $\Omega^i$ extracted from the relevant grade component of $\Delta_*|_{\Omega^i}$. For example, $\Delta_*|_{\Omega^1} = \Delta_L + \Delta_R$ for the coactions on $\Omega^1$.

Finally, in the strongly bicovariant case it follows [39] by a super version of the Radford-Majid biproduct (see Lemma 2.4.1) that $\Omega \cong A{\triangleright\!\!\!<}\Lambda$ is a super-bosonisation, where $\Lambda = \oplus_{i\geq 1}\Lambda^i$ is the subalgebra of left-invariant differential forms and forms a super braided Hopf algebra in category of right $A$-crossed modules $\mathcal{M}_A^A$.

**Lemma 2.5.7.** *Let $A, H$ be Hopf algebras, and let $\Omega(A), \Omega(H)$ be strongly bicovariant exterior algebras. Then $\Omega(A \otimes H) := \Omega(A)\underline{\otimes}\Omega(H)$ is a strongly bicovariant exterior algebra* *with super tensor product algebra and super tensor coproduct coalgebra, and differential*

$$\mathrm{d}(\omega\otimes\eta)=\mathrm{d}_A\omega\otimes\eta+(-1)^{|\omega|}\omega\otimes\mathrm{d}_H\eta,$$

*for all* $\omega,\tau\in\Omega(A)$, $\eta,\xi\in\Omega(H)$. *Furthermore, it is isomorphic to* $(A\otimes H)\bowtie(\Lambda_A\otimes\Lambda_H)$.

*Proof.* It is easy to see that $\Omega^1(A\otimes H)=\mathrm{span}\{(a\otimes h)\mathrm{d}(b\otimes g)\}=\mathrm{span}\{a\mathrm{d}_A b\otimes f+c\otimes h\mathrm{d}_H g\}=\Omega^1(A)\otimes H\oplus A\otimes\Omega(H)$ since

$$(c\otimes h)\mathrm{d}(1\otimes g)=c\otimes h\mathrm{d}_H g,\quad (a\otimes 1)\mathrm{d}(b\otimes f)-(ab\otimes 1)\mathrm{d}(1\otimes f)=a\mathrm{d}_A b\otimes f$$

for all $a,b,c\in A$ and $h,g,f\in H$. The graded Leibniz-rule holds since $\mathrm{d}(\omega\otimes 1)=\mathrm{d}_A\omega$ and $\mathrm{d}(1\otimes\eta)=\mathrm{d}_H\eta$ for all $\omega\in\Omega(A)$, $\eta\in\Omega(A)$, and the algebra is just a super tensor product algebra. The new part is that d is a super-coderivation :

$$\begin{aligned}\Delta_*\mathrm{d}(\omega\eta)=&\Delta_*((\mathrm{d}\omega)\eta+(-1)^{|\omega|}\omega\mathrm{d}\eta)\\
=&(\Delta_*\mathrm{d}\omega)\Delta_*\eta+(-1)^{|\omega|}(\Delta_*\omega)(\Delta_*\mathrm{d}\eta)\\
=&((\mathrm{d}\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*\omega)(\Delta_*\eta)+(-1)^{|\omega|}(\Delta_*\omega)((\mathrm{d}\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*\eta)\\
=&(-1)^{|\eta_{(1)}||\omega_{(2)}|}\Big(\big((\mathrm{d}\omega_{(1)})\eta_{(1)}+(-1)^{|\omega_{(1)}|}\omega_{(1)}\mathrm{d}\eta_{(1)}\big)\otimes\omega_{(2)}\eta_{(2)}\\
&+(-1)^{|\omega_{(1)}|+|\eta_{(1)}|}\omega_{(1)}\eta_{(1)}\otimes\big((\mathrm{d}\omega_{(2)})\eta_{(2)}+(-1)^{|\omega_{(2)}|}\omega_{(2)}\mathrm{d}\eta_{(2)}\big)\Big)\\
=&(-1)^{|\eta_{(1)}||\omega_{(2)}|}(\mathrm{d}(\omega_{(1)}\eta_{(1)})\otimes\omega_{(2)}\eta_{(2)}+(-1)^{|\omega_{(1)}\eta_{(1)}|}\omega_{(1)}\eta_{(1)}\otimes\omega_{(2)}\eta_{(2)})\\
=&(\mathrm{d}\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*(\omega\eta).\end{aligned}$$

Furthermore, $A\otimes H$ acts and coacts on $\Lambda_A\otimes\Lambda_H$ by $(v\otimes w)\triangleleft(a\otimes h)=v\triangleleft a\otimes w\triangleleft h$ and $\Delta_R(v\otimes w)=v^{\overline{(0)}}\otimes w^{\overline{(0)}}\otimes v^{\overline{(1)}}\otimes w^{\overline{(1)}}$ for all $a\in A,h\in H$ and $v\in\Lambda_A,w\in\Lambda_H$, making $\Lambda_A\otimes\Lambda_H$ an $A\otimes H$-crossed module since

$$\begin{aligned}\Delta_R((v\otimes w)\triangleleft(a\otimes h))=&(v^{\overline{(0)}}\otimes w^{\overline{(0)}})\triangleleft(a_{(2)}\otimes h_{(2)})\otimes(S(a_{(1)}\otimes h_{(1)}))(v^{\overline{(1)}}\otimes w^{\overline{(1)}})(a_{(3)}\otimes h_{(3)})\\
=&(v^{\overline{(0)}}\triangleleft a_{(2)})\otimes(w^{\overline{(0)}}\triangleleft h_{(2)})\otimes(Sa_{(1)})v^{\overline{(1)}}a_{(3)}\otimes(Sh_{(1)})w^{\overline{(1)}}h_{(3)}\end{aligned}$$

$$=\Delta_R((v{\triangleleft}a)\otimes(w{\triangleleft}h)).$$

Thus there is a bosonisation $(A\otimes H){\triangleright\!\!\!<}(\Lambda_A\otimes\Lambda_H)$ and it is easy to show that $\Omega(A)\otimes\Omega(H)\cong A{\triangleright\!\!\!<}\Lambda_A\otimes H{\triangleright\!\!\!<}\Lambda_H\cong(A\otimes H){\triangleright\!\!\!<}(\Lambda_A\otimes\Lambda_H)$ by $a\otimes v\otimes h\otimes w\mapsto a\otimes h\otimes v\otimes w$. $\square$

# Chapter 3

# (Co)double bosonisation and dual basis of $c_q[SL_2]$

In [28], Majid introduced a construction of quasitriangular Hopf algebra $C^{\underline{\mathrm{cop}}}{>\!\!\!\triangleleft} H {\triangleright\!\!\!<} B$ called double bosonisation. Here $H$ is a quasitriangular Hopf algebra, $B$ is a braided Hopf algebra living in $\mathcal{M}_H$, and $C = B^{\star \mathrm{op/cop}}$, where $B^\star$ is a categorical dual to $B$. One can think of $H$ as Cartan subalgebra, $B$ as positive root space, and $C$ as negative root space. Moreover, Majid proved that quantum groups $U_q(\mathfrak{g})$ can be constructed using double bosonisation. Justifying its terminology, double bosonisation contains bosonisations $H{\triangleright\!\!\!<} B$ and $C^{\underline{\mathrm{cop}}}{>\!\!\!\triangleleft} H$ as sub-Hopf algebra, which gives a positive and negative Borel subalgebra in the case of $U_q(\mathfrak{g})$.

We work out the dual version of double bosonisation in Section 3.2. In doing so, we need to restrict our case to finite-dimensional braided Hopf algebra so that $C = B^*$ becomes the ordinary dual of $B$. We then find a coquasitriangular Hopf algebra $B^{\underline{\mathrm{op}}}{>\!\!\!\triangleleft} A {\triangleright\!\!\!<} B^*$ associated to a finite-dimensional Hopf algebra $B$ in ${}^A\mathcal{M}$, where $A$ is a coquasitriangular Hopf algebra. Here the product and coproduct of codouble bosonisation are obtained by taking the duality pairing with the coproduct and product on double bosonisation. However, we do not want to be limited to finite-dimensional $A$ and give a direct proof for the codouble bosonisation as a coquasitriangular Hopf algebra. Also note that unlike double bosonisation, the codouble bosonisation has coalgebra surjections to its components $B^{\underline{\mathrm{op}}}, A, B^*$, making calculations on the codouble bosonisation harder than the original version.

In Section 3.3, we take $H = \mathbb{C}[K]/(K^n - 1)$ for any $n$ with its natural quasitriangular structure $\mathcal{R}_K$ such that its modules are the category of $\mathbb{Z}_n$-graded spaces with braiding given by a power of $q$ according to the degrees, where $q$ is a primitive $n$-th root of unity. We take $B = \mathbb{C}[E]/(E^n)$ and $B^{*\underline{\mathrm{cop}}} = \mathbb{C}[F]/(F^n)$ and by double bosonisation, we will obtain

$$B^{*\underline{\mathrm{cop}}} \rtimes\!\!\!\lhd H \rhd\!\!\!\ltimes B = \mathfrak{u}_q(sl_2) \cong \begin{cases} u_p(sl_2) & n = 2m+1,\ p = q^{-m};\ p^2 = q, \\ \text{something else} & n \text{ even}. \end{cases}$$

To illustrate the even case, $\mathfrak{u}_{-1}(sl_2)$ in Example 3.3.4 is an interesting 8-dimensional strictly quasitriangular and self-dual Hopf algebra presumably known elsewhere. Similarly, we take $B = \mathbb{C}[x]/(x^n)$ and its dual $B^* = \mathbb{C}[Y]/(Y^n)$, again braided-lines but this time viewed in the category of $A = \mathbb{C}[t]/(t^n - 1)$-comodules with its standard coquasitriangular structure $\mathcal{R}(t,t) = q$ (so that its comodules form the same braided category of $\mathbb{Z}_n$-graded vector spaces as before). Then the codouble bosonisation will give a coquasitriangular Hopf algebra $\mathfrak{c}_q[SL_2] = c_p[SL_2]$ when $n$ is odd and some other coquasitriangular Hopf algebra when $n$ is even. This is Theorem 3.3.1 with the dual basis result a corollary of the triangular decomposition. As an application, we work out a Hopf algebra Fourier transform $\mathcal{F} : \mathfrak{c}_q[SL_2] \to \mathfrak{u}_q(sl_2)$ in Section 3.4, and we will see that it is well behaved with respect to the 3D-calculus of $c_p[SL_2]$.

## 3.1 Double Bosonisation

We first need to recall basic facts about braided Hopf algebras which will be useful for the construction of the double bosonisation. If $B \in \mathcal{C}$ is a braided Hopf algebra with invertible antipode in the braided monoidal category $\mathcal{C}$, then $B^{\underline{\mathrm{cop}}}$ with the same algebra structure as $B$ but with braided coopposite coproduct and antipode given by

$$\underline{\Delta}_{\underline{\mathrm{cop}}} = \Psi^{-1}_{B,B} \circ \underline{\Delta}, \quad \overline{S} = \underline{S}^{-1} \tag{3.1.1}$$

is a braided Hopf algebra in $\overline{\mathcal{C}}$, by which we mean $\mathcal{C}$ with the reversed (inverse) braid crossing [24]. In a concrete setting, we write $\underline{\Delta}_{\underline{\mathrm{cop}}} c = c^{\overline{(1)}} \otimes c^{\overline{(2)}}$ for all $c \in B^{\underline{\mathrm{cop}}}$ (summation understood).

Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra with invertible antipode, and let $B$ be a finite-dimensional braided Hopf algebra in $\mathcal{M}_H$, and let $B^*$ be an ordinary dual of $B$ living in ${}_H\mathcal{M}$ as explained in Lemma 2.4.2. We have braided Hopf algebra $B^{*\underline{\mathrm{cop}}}$ in ${}_{\overline{H}}\mathcal{M}$. Thus, by bosonisation we have two Hopf algebras $H{\triangleright\!\!<}B$ and $B^{*\underline{\mathrm{cop}}}{>\!\!\triangleleft}\overline{H}$. We can glue them together to get the following theorem of double bosonisation.

**Theorem 3.1.1.** *c.f. [28, Theorem 3.2] Let $H$ be a quasitriangular Hopf algebra. Let $B$ be a finite-dimensional braided Hopf algebra in $\mathcal{M}_H$. There is an ordinary Hopf algebra $D_H(B) = B^{*\underline{\mathrm{cop}}}{>\!\!\triangleleft}H{\triangleright\!\!<}B$, the double bosonisation, built on $B^{*\underline{\mathrm{cop}}} \otimes H \otimes B$ with product and coproduct*

$$\begin{aligned}(c \otimes h \otimes b)(d \otimes g \otimes a) =& c((h_{(1)}\mathcal{R}_1^{(2)}){\triangleright} d_{\overline{(2)}}) \otimes h_{(2)}\mathcal{R}_2^{(2)}\mathcal{R}_1^{-(1)}g_{(1)} \otimes (b_{\underline{(2)}}{\triangleleft}(\mathcal{R}_2^{-(1)}g_{(2)}))a \\ & \langle \mathcal{R}_1^{(1)}{\triangleright} d_{\overline{(1)}}, b_{\underline{(1)}}{\triangleleft}\mathcal{R}_2^{(1)}\rangle\langle \mathcal{R}_1^{-(2)}{\triangleright}\overline{S}d_{\overline{(3)}}, b_{\underline{(3)}}{\triangleleft}\mathcal{R}_2^{-(2)}\rangle \\ \Delta(c \otimes h \otimes b) =& c_{\overline{(1)}} \otimes \mathcal{R}^{-(1)}h_{(1)} \otimes b_{\underline{(1)}}{\triangleleft}\mathcal{R}^{(1)} \otimes \mathcal{R}^{-(2)}{\triangleright} c_{\overline{(2)}} \otimes h_{(2)}\mathcal{R}^{(2)} \otimes b_{\underline{(2)}}\end{aligned}$$

*for all $a, b \in B$, $c, d \in B^*$ and $h, g \in H$, where $\mathcal{R}_1, \mathcal{R}_2$ are copies of $\mathcal{R}$. Furthermore, $B^{*\underline{\mathrm{cop}}}{>\!\!\triangleleft}H{\triangleright\!\!<}B$ has a quasitriangular structure*

$$\mathcal{R}_{\mathrm{new}} = \overline{\exp} \cdot \mathcal{R}, \quad \overline{\exp} = \sum f^a \otimes \underline{S}e_a$$

*where $\{e_a\}$ is a basis of $B$ and $\{f^a\}$ is a dual basis of $B^*$.*

In the original theory, $B$ is not required to be finite-dimensional, but we have restricted to the finite-dimensional case for simplicity. More generally, one has a double of biproduct Hopf algebra when we consider braided Hopf algebra $B$ lives in crossed module category $\mathcal{M}_H^H$, but this is beyond our scope.

## 3.2 Codouble Bosonisation

Our goal in this section is to find a dual version of the double bosonisation theorem with $A$ coquasitriangular and $B \in {}^A\mathcal{M}$, in which case the category with reversed braiding is ${}^{\overline{A}}\mathcal{M}$ and

$$a \cdot_{\underline{\mathrm{op}}} b = \mathcal{R}(Sa^{\overline{(1)}}, b^{\overline{(1)}}) b^{\overline{(\infty)}} a^{\overline{(\infty)}} \tag{3.2.1}$$

for all $a, b \in B^{\underline{\mathrm{op}}}$. As in Lemma 2.4.3, we think of ${}^A\mathcal{M}$ as $\mathcal{M}_H$ in the finite-dimensional Hopf algebra case by evaluating against a coaction of $A$ to get an action of $H$.

**Lemma 3.2.1.** *If $H$ is finite-dimensional and quasitriangular with dual $A$ and $B \in {}^A\mathcal{M}$ is finite-dimensional then $(B^{\underline{\mathrm{op}}})^* = B^{*\underline{\mathrm{cop}}} \in {}_{\overline{H}}\mathcal{M}$.*

*Proof.* Here $B^{\underline{\mathrm{op}}} \in {}^{\overline{A}}\mathcal{M}$ or $\mathcal{M}_{\overline{H}}$ and $(B^{\underline{\mathrm{op}}})^* \in {}_{\overline{H}}\mathcal{M}$ where $B^{*\underline{\mathrm{cop}}}$ lives. It is clear that the coproduct of $B^{\underline{\mathrm{op}}}$ corresponds to the product of $B^{*\underline{\mathrm{cop}}}$. For the other half,

$$\begin{aligned}\langle x, b \cdot_{\underline{\mathrm{op}}} c\rangle &= \langle x, c^{\overline{(\infty)}} b^{\overline{(\infty)}}\rangle \mathcal{R}(Sb^{\overline{(1)}}, c^{\overline{(1)}}) = \langle x_{\underline{(1)}}, c^{\overline{(\infty)}}\rangle\langle x_{\underline{(2)}}, b^{\overline{(\infty)}}\rangle\langle b^{\overline{(\infty)}}, \mathcal{R}^{-(2)}\rangle\langle c^{\overline{(1)}}, \mathcal{R}^{-(1)}\rangle \\ &= \langle x_{\underline{(1)}}, c \triangleleft \mathcal{R}^{-(1)}\rangle\langle x_{\underline{(2)}}, b \triangleleft \mathcal{R}^{-(2)}\rangle = \langle \mathcal{R}^{-(2)} \triangleright x_{\underline{(2)}} \otimes \mathcal{R}^{-(1)} \triangleright x_{\underline{(1)}}, b \otimes c\rangle\end{aligned}$$

which is $\langle \underline{\Delta}_{\underline{\mathrm{cop}}} x, b \otimes c\rangle$ as required. □

The dual version of Theorem 3.1.1 can in principle now be deduced at least when $A$ is finite-dimensional. However, we do not want to be limited to this case and give a direct proof of the resulting formulae.

**Theorem 3.2.2** (Codouble bosonisation)**.** *Let $B$ be a finite-dimensional braided Hopf algebra in ${}^A\mathcal{M}$ with basis $\{e_a\}$. Denote its dual by $B^* \in \mathcal{M}^A$ with dual basis $\{f^a\}$. Then there is an ordinary Hopf algebra $B^{\underline{\mathrm{op}}} {\bowtie\!\!\!\triangleleft} A {\triangleright\!\!\!\bowtie} B^*$, the* codouble bosonisation*, built on the* *vector space $B^{\underline{\mathrm{op}}} \otimes A \otimes B^*$ with*

$$(x \otimes k \otimes y)(w \otimes \ell \otimes z) = x \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}} \otimes k_{(2)}\ell_{(1)} \otimes y^{\overline{(0)}} z \mathcal{R}(y^{\overline{(1)}}, \ell_{(2)}) \mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}),$$

$$\begin{aligned}\Delta(x \otimes k \otimes y) \\ = \sum_a x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} k_{(1)} \otimes f^a \otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{a\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(4)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}} \\ \mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} k_{(2)}) \mathcal{R}(S(k_{(3)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}}) \ \langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}} \rangle\end{aligned}$$

*for all $x, w \in B^{\underline{\mathrm{op}}}$, $k, \ell \in A$, and $y, z \in B^*$.*

Here $B^{\underline{\mathrm{op}}}$, $A$ and $B^*$ are subalgebras of $B^{\underline{\mathrm{op}}} {>\!\!\!\triangleleft} A {\triangleright\!\!\!<} B^*$ and identifying $x \equiv x \otimes 1 \otimes 1$, $k \equiv 1 \otimes k \otimes 1$ and $y \equiv 1 \otimes 1 \otimes y$ we have $xky \equiv x \otimes k \otimes y$. We also have algebra maps

$$B^* \hookrightarrow B^{\underline{\mathrm{op}}} {>\!\!\!\triangleleft} A {\triangleright\!\!\!<} B^* \twoheadrightarrow B^{\underline{\mathrm{op}}} {>\!\!\!\triangleleft} A, \quad B^{\underline{\mathrm{op}}} \hookrightarrow B^{\underline{\mathrm{op}}} {>\!\!\!\triangleleft} A {\triangleright\!\!\!<} B^* \twoheadrightarrow A {\triangleright\!\!\!<} B^*$$

where the surjections are $\mathrm{id} \otimes \underline{\epsilon}$ and $\underline{\epsilon} \otimes \mathrm{id}$ respectively. It remains to prove Theorem 3.2.2, which we will prove by the following separate lemmas.

**Lemma 3.2.3.** *The product stated in Theorem 3.2.2 is associative.*

*Proof.* We expand the definition of the product to find

$$\begin{aligned}&\Big((x \otimes k \otimes y)(w \otimes \ell \otimes z)\Big)(m \otimes j \otimes v) \\ &= (x \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}} \otimes k_{(2)}\ell_{(1)} \otimes y^{\overline{(0)}} z)(m \otimes j \otimes v) \ \mathcal{R}(y^{\overline{(1)}}, \ell_{(2)}) \mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}) \\ &= x \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} m^{\overline{(\infty)}} \otimes k_{(3)}\ell_{(2)} j_{(1)} \otimes y^{\overline{(0)}\,\overline{(0)}} z^{\overline{(0)}} v \ \mathcal{R}(y^{\overline{(1)}}, \ell_{(3)}) \mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}) \\ &\quad \mathcal{R}(y^{\overline{(0)}\,\overline{(1)}} z^{\overline{(1)}}, j_{(2)}) \mathcal{R}(S(k_{(2)}\ell_{(1)}), m^{\overline{(1)}}) \\ &= x \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} m^{\overline{(\infty)}} \otimes k_{(3)}\ell_{(2)} j_{(1)} \otimes y^{\overline{(0)}} z^{\overline{(0)}} v \ \mathcal{R}(y^{\overline{(1)}}{}_{(2)}, \ell_{(3)}) \mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}) \\ &\quad \mathcal{R}(y^{\overline{(1)}}{}_{(1)}, j_{(2)}) \mathcal{R}(z^{\overline{(1)}}, j_{(3)}) \mathcal{R}(S\ell_{(1)}, m^{\overline{(1)}}{}_{(1)}) \mathcal{R}(Sk_{(2)}, m^{\overline{(1)}}{}_{(2)}),\end{aligned}$$

for all $x, w, m \in B^{\underline{\mathrm{op}}}$, $k, \ell, j \in A$ and $y, z, v \in B^*$, where the last equality uses the right-coaction property on $y$. Similarly,

$$
\begin{aligned}
&(x{\otimes}k \otimes y)\Big((w \otimes \ell \otimes z)(m \otimes j \otimes v)\Big)\\
&\quad=(x \otimes k \otimes y)(w \cdot_{\underline{\mathrm{op}}} m^{\overline{(\infty)}} \otimes \ell_{(2)}j_{(1)} \otimes z^{\overline{(0)}}v)\ \mathcal{R}(z^{\overline{(1)}}, j_{(2)})\mathcal{R}(S\ell_{(1)}, m^{\overline{(1)}})\\
&\quad=x \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} m^{\overline{(\infty)}\,\overline{(\infty)}} \otimes k_{(2)}\ell_{(2)}j_{(1)} \otimes y^{\overline{(0)}}z^{\overline{(0)}}v\ \mathcal{R}(z^{\overline{(1)}}, j_{(3)})\mathcal{R}(S\ell_{(1)}, m^{\overline{(1)}})\\
&\qquad\mathcal{R}(y^{\overline{(1)}}, \ell_{(3)}j_{(2)})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}m^{\overline{(\infty)}\,\overline{(1)}}),
\end{aligned}
$$

which by the left-coaction property on $m$ agrees with our first calculation. □

**Lemma 3.2.4.** *The coproduct $\Delta$ stated in Theorem 3.2.2 is an algebra map.*

*Proof.* Expanding the product and then the coproduct, we have

$$
\begin{aligned}
&\Delta\Big((x \otimes k \otimes y)(w \otimes \ell \otimes z)\Big)\\
&=x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}}{}_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(\infty)}\,\overline{(1)}}{}_{(1)}w^{\overline{(\infty)}}{}_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(2)}\ell_{(1)} \otimes f^a\\
&\quad\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}\,\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}}{}_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(5)}\ell_{(4)}y^{\overline{(0)}}{}_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}\,\overline{(1)}}{}_{(2)}\\
&\quad\otimes y^{\overline{(0)}}{}_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\ \mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(\infty)}\,\overline{(1)}}{}_{(2)}w^{\overline{(\infty)}}{}_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(3)}\ell_{(2)})\\
&\quad\mathcal{R}(S(k_{(4)}\ell_{(3)}y^{\overline{(0)}}{}_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}z_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}})\mathcal{R}(y^{\overline{(1)}}, \ell_{(5)})\\
&\quad\mathcal{R}(Sx_{\underline{(2)}}{}^{\overline{(1)}}, w^{\overline{(\infty)}}{}_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(y^{\overline{(0)}}{}_{\underline{(2)}}{}^{\overline{(1)}}, z_{\underline{(1)}}{}^{\overline{(1)}})\langle y^{\overline{(0)}}{}_{\underline{(1)}}{}^{\overline{(0)}}z_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(0)}}, e_{a\underline{(2)}}\rangle\\
&=x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}\ell_{(1)} \otimes f^a\\
&\quad\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}} \otimes k_{(5)}\ell_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)}\ell_{(2)})\mathcal{R}(Sk_{(1)}, w_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}, z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\\
&\quad\mathcal{R}(S(k_{(4)}\ell_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(4)}}{}^{\overline{(1)}})\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}, \ell_{(5)})\mathcal{R}(Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}, w_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\\
&\quad\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(3)}}\rangle\\
&=x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(3)}\ell_{(1)} \otimes f^a\\
&\quad\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}} \otimes k_{(6)}\ell_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(4)}\ell_{(2)})\mathcal{R}(Sk_{(2)}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}), w_{\underline{(1)}}{}^{\overline{(1)}})\\
&\quad\mathcal{R}(S(k_{(5)}\ell_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(4)}}{}^{\overline{(1)}})\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}, \ell_{(5)})\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}}, \ell_{(6)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})
\end{aligned}
$$

$$\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(3)}}\rangle$$

for all $x, w \in B^{\underline{\mathrm{op}}}$, $k, \ell \in A$, and $y, z \in B^*$. The second equality uses the comodule coalgebra property (2.1.8) on $w$ and coassociativity. The last expression uses coquasitriangularity (2.2.4) to gather the parts of $w_{\underline{(1)}}{}^{\overline{(1)}}$ and $y_{\underline{(2)}}{}^{\overline{(1)}}$ inside $\mathcal{R}$. On the other side,

$$\begin{aligned}
&\Delta(x \otimes k \otimes y)\Delta(w \otimes \ell \otimes z)\\
=&x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)(2)}k_{(1)(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)(1)}\ell_{(1)(1)} \otimes f^{a\overline{(0)}} f^b\\
&\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\mathrm{op}} e_{b\underline{(1)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\mathrm{op}} \overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}\overline{(\infty)}}\\
&\otimes k_{(4)(2)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)(2)}\ell_{(4)(1)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)(1)} \otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\\
&\mathcal{R}(S(k_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}})\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(2)})\mathcal{R}(S(\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(3)}}{}^{\overline{(1)}})\\
&\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)(1)}k_{(1)(1)}), w_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(f^{a\overline{(1)}}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)(2)}\ell_{(1)(2)})\\
&\mathcal{R}(S(k_{(4)(1)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)(1)}), e_{b\underline{(1)}}{}^{\overline{(\infty)}\overline{(1)}}w_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(1)}}e_{b\underline{(3)}}{}^{\overline{(\infty)}\overline{(1)}})\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}}, \ell_{(4)(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)(2)})\\
&\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(2)}}\rangle\\
=&x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}\ell_{(1)} \otimes f^a f^b\\
&\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{a\underline{(4)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(6)}}{}^{\overline{(\infty)}\overline{(\infty)}}\\
&\otimes k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}\ell_{(5)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\ \mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(\infty)}\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\\
&\mathcal{R}(S(k_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(\infty)}\overline{(1)}})\mathcal{R}(e_{a\underline{(4)}}{}^{\overline{(1)}}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}\ell_{(3)})\mathcal{R}(S(\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(6)}}{}^{\overline{(1)}})\\
&\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}), w_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}e_{a\underline{(2)}}{}^{\overline{(1)}}e_{a\underline{(3)}}{}^{\overline{(1)}}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(2)})\\
&\mathcal{R}(S(k_{(5)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}), e_{a\underline{(4)}}{}^{\overline{(\infty)}\overline{(1)}}w_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(1)}}e_{a\underline{(6)}}{}^{\overline{(\infty)}\overline{(1)}})\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}}, \ell_{(6)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\\
&\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}{}^{\overline{(\infty)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(5)}}\rangle\\
=&x_{\underline{(1)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}\ell_{(1)} \otimes f^a\\
&\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}} \otimes k_{(5)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{\underline{(2)}}\\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\mathcal{R}(S(\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}), w_{\underline{(1)}}{}^{\overline{(1)}})\\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(2)})\mathcal{R}(S(k_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}), w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(2)})
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}},\ell_{(5)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\langle y_{\underline{(1)}}{}^{\overline{(0)}},e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}},e_{a\underline{(3)}}\rangle\\
=&x_{\underline{(1)}}\cdot_{\underline{\mathrm{op}}} w_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}\ell_{(1)}\otimes f^a\\
&\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} w_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\mathrm{op}}\overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}}\otimes k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}\ell_{(5)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y_{\underline{(2)}}{}^{\overline{(0)}}z_{(2)}\\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)}w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(2)})\mathcal{R}(S(k_{(5)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),e_{a\underline{(4)}}{}^{\overline{(1)}})\\
&\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}),w_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(y_{\underline{(2)}}{}^{\overline{(1)}},\ell_{(6)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},\ell_{(3)})\mathcal{R}(Sk_{(4)},w_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\\
&\langle y_{\underline{(1)}}{}^{\overline{(0)}},e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}},e_{a\underline{(3)}}\rangle
\end{aligned}
$$

where the second equality uses duality $\langle f^{a\overline{(0)}},e_a\rangle f^{a\overline{(1)}}=\langle f^a,e_a^{\overline{(\infty)}}\rangle e_a^{\overline{(1)}}$ followed by the comodule coalgebra property (2.1.8) on $e_a$. The third equality cancels $(\overline{S}e_{a\underline{(3)}}\cdot_{\underline{\mathrm{op}}}e_{a\underline{(4)}})^{\overline{(\infty)}\overline{(\infty)}}$ making all subsequent coactions trivial. The fourth equality uses (2.2.4) to gather the parts of $e_{a\underline{(1)}}{}^{\overline{(1)}}$ and $e_{a\underline{(4)}}{}^{\overline{(1)}}$ inside $\mathcal{R}$, and cancels some $\mathcal{R}$s. In the final expression, one can use quasicommutativity (2.2.5) to reorder the second tensor factor so as to coincide with the result of the first calculation. □

**Lemma 3.2.5.** *The coproduct $\Delta$ stated in Theorem 3.2.2 is coassociative.*

*Proof.* We expand the definition of the coproduct to find

$$
\begin{aligned}
&(\mathrm{id}\otimes\Delta)\Delta(x\otimes k\otimes y)\\
=&x_{\underline{(1)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes f^a\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(5)}}{}^{\overline{(\infty)}}\\
&\otimes e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(3)}k_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes f^b\\
&\otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} e_{a\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}}\otimes k_{(7)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\otimes y_{\underline{(3)}}\\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}},e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(4)}k_{(5)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S(k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}),e_{b\underline{(3)}}{}^{\overline{(1)}})\\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(k_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(1)}e_{a\underline{(5)}}{}^{\overline{(1)}}{}_{(1)})\\
&\mathcal{R}(S(e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}),e_{a\underline{(5)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(Se_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(2)},e_{a\underline{(5)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(Se_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(2)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\\
&\langle y_{\underline{(1)}}{}^{\overline{(0)}},e_{a\underline{(3)}}\rangle\langle y_{\underline{(2)}}{}^{\overline{(0)}},e_{b\underline{(2)}}\rangle\\
=&x_{\underline{(1)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes f^a\otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(5)}}{}^{\overline{(\infty)}}\\
&\otimes e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(3)}k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}\otimes f^b
\end{aligned}
$$

$$
\begin{aligned}
&\otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{a\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(9)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(6)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(3)}} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(6)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(4)}k_{(7)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(S(k_{(8)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(3)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(2)}k_{(5)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}), e_{a\underline{(5)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\mathcal{R}(S(k_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(1)})\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(3)}}\rangle\langle y_{\underline{(2)}}{}^{\overline{(0)}}, e_{b\underline{(2)}}\rangle \\
=&x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)} \otimes f^a \otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(5)}}{}^{\overline{(\infty)}} \\
&\otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(5)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)} \otimes f^b \\
&\otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{a\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(4)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(9)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(6)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(3)}} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}k_{(6)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(S(k_{(8)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(3)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(4)}e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(2)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}), e_{a\underline{(5)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}, k_{(3)})\mathcal{R}(e_{a\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(6)})\mathcal{R}(Sy_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(Sk_{(7)}, e_{a\underline{(4)}}{}^{\overline{(1)}}{}_{(5)}) \\
&\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(3)}}\rangle\langle y_{\underline{(2)}}{}^{\overline{(0)}}, e_{b\underline{(2)}}\rangle,
\end{aligned}
$$

where the second equality uses (2.2.4) to gather the parts of $e_{a\underline{(1)}}{}^{\overline{(1)}}$ and $e_{a\underline{(5)}}{}^{\overline{(1)}}$, cancelling some of the $\mathcal{R}$s. We lastly use (2.2.5) to change the order in the fifth tensor factor and in a similar term inside $\mathcal{R}$, again cancelling some of the $\mathcal{R}$s. On the other side,

$$
\begin{aligned}
&(\Delta \otimes \mathrm{id})\Delta(x \otimes k \otimes y) \\
=&x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)} \otimes f^b \otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}k_{(4)}f^a{}_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \\
&\otimes f^a{}_{\underline{(2)}} \otimes e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(7)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)}f^a{}_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(3)}}{}^{\overline{(1)}})\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}k_{(5)}) \\
&\mathcal{R}(S(k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}})\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\langle f^a{}_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(2)}}\rangle \\
=&x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)} \otimes f^b \otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}} \otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}k_{(4)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} \\
&\otimes f^c \otimes (e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}})_{\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\mathrm{op}} x_{\underline{(3)}}{}^{\overline{(\infty)}}\overline{S}(e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}})_{\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(7)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{\underline{(2)}} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(3)}}{}^{\overline{(1)}}) \\
&\mathcal{R}((e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}})_{\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}k_{(5)})\mathcal{R}(S(k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), (e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}})_{\underline{(3)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_c^{\overline{(1)}}, e_a^{\overline{(1)}})\langle f^a, e_{b\underline{(2)}}{}^{\overline{(\infty)}}\rangle\langle y_{\underline{(1)}}{}^{\overline{(0)}}, (e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}})_{\underline{(2)}}\rangle
\end{aligned}
$$

$$
\begin{aligned}
=&x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)} k_{(1)} \otimes f^b \otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{b\underline{(5)}}{}^{\overline{(\infty)}} \\
&\otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)} k_{(4)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(2)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(2)} \otimes f^c \\
&\otimes e_{c\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}(e_{c\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(4)}}{}^{\overline{(\infty)}}) \otimes k_{(7)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes y_{(2)} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)} k_{(2)}) \mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)} k_{(3)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(1)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(5)}}{}^{\overline{(1)}}) \\
&\mathcal{R}(e_{c\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(6)}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)} k_{(5)}) \mathcal{R}(S(k_{(6)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(3)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(4)}) \\
&\mathcal{R}(e_{c\underline{(1)}}{}^{\overline{(1)}}{}_{(1)} e_{c\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} e_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}, e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(3)} e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(3)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(3)}) \mathcal{R}(S e_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(2)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(4)} e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}) \\
&\mathcal{R}(S e_{c\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}, e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}) \langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{c\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(3)}}{}^{\overline{(\infty)}} \rangle.
\end{aligned}
$$

For the second equality we use duality $\langle f^a{}_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(2)}} \rangle f^a{}_{\underline{(1)}}{}^{\overline{(1)}} = \langle f^a{}_{\underline{(1)}}, e_{b\underline{(2)}}{}^{\overline{(\infty)}} \rangle e_{b\underline{(2)}}{}^{\overline{(1)}}$ to replace $f^a{}_{\underline{(1)}}{}^{\overline{(1)}}$ by $e_{b\underline{(2)}}{}^{\overline{(1)}}$, followed by

$$
e_a \otimes f^a{}_{\underline{(1)}} \otimes f^a{}_{\underline{(2)}} = e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_a^{\overline{(\infty)}} \otimes f^a \otimes f^c \; \mathcal{R}(e_c^{\overline{(1)}}, e_a^{\overline{(1)}})
$$

to replace $f^a{}_{\underline{(1)}} \otimes f^a{}_{\underline{(2)}}$ by $f^a \otimes f^c$. For the third equality, we use $\langle f^a, e_{b\underline{(2)}}{}^{\overline{(\infty)}} \rangle$ to replace $e_a$ by $e_{b\underline{(2)}}{}^{\overline{(\infty)}}$, after which we expand $(e_c^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(2)}}{}^{\overline{(\infty)}\overline{(\infty)}})_{\underline{(1)}}$ etc. using $\underline{\Delta}$ a braided-homomorphism. In the last expression, we expand $\overline{S}$ of a $\cdot_{\underline{\mathrm{op}}}$ product and use

$$
\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{c\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(3)}}{}^{\overline{(\infty)}} \rangle = \langle y_{\underline{(1)}}{}^{\overline{(0)}}{}_{\underline{(1)}}, e_{b\underline{(3)}}{}^{\overline{(\infty)}} \rangle \langle y_{\underline{(1)}}{}^{\overline{(0)}}{}_{\underline{(2)}}, e_{c\underline{(2)}}{}^{\overline{(\infty)}} \rangle \mathcal{R}(S e_{c\underline{(2)}}{}^{\overline{(\infty)}\overline{(1)}}, e_{b\underline{(3)}}{}^{\overline{(\infty)}\overline{(1)}}).
$$

By the comodule coalgebra property (2.1.7), the first pairing on the right becomes $\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(3)}}{}^{\overline{(\infty)}} \rangle \langle y_{\underline{(2)}}{}^{\overline{(0)}}, e_{c\underline{(2)}}{}^{\overline{(\infty)}} \rangle$ and duality $\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(3)}}{}^{\overline{(\infty)}} \rangle e_{b\underline{(3)}}{}^{\overline{(1)}} = \langle y_{\underline{(1)}}{}^{\overline{(0)}\overline{(0)}}, e_{b\underline{(3)}} \rangle y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}$ replaces $e_{b\underline{(3)}}{}^{\overline{(1)}}$ by $y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}$. The other pairing similarly replaces $e_{c\underline{(2)}}{}^{\overline{(1)}}$ by $y_{\underline{(2)}}{}^{\overline{(0)}\overline{(1)}}$, so

$$
\begin{aligned}
&(\Delta \otimes \mathrm{id})\Delta(x \otimes k \otimes y) \\
=&x_{\underline{(1)}} \otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)} k_{(1)} \otimes f^b \otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\mathrm{op}} \overline{S} e_{b\underline{(5)}}{}^{\overline{(\infty)}} \\
&\otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)} k_{(4)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(2)} \otimes f^c \\
&\otimes e_{c\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_{b\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{b\underline{(4)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{c\underline{(3)}}{}^{\overline{(\infty)}} \otimes k_{(7)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(7)} y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)} \otimes y_{(3)} \\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)} k_{(2)}) \mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)} k_{(3)} e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)} e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(5)}}{}^{\overline{(1)}})
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{R}(e_{c\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(6)}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}k_{(5)})\mathcal{R}(S(k_{(6)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(6)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}), e_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(4)})\\
&\mathcal{R}(Se_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}, e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(e_{c\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}e_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}, e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(3)})\\
&\mathcal{R}(Sy_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}, e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(5)})\mathcal{R}(Sy_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}, y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)})\mathcal{R}(Se_{c\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}, e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(5)})\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(3)}}\rangle\langle y_{\underline{(2)}}{}^{\overline{(0)}}, e_{c\underline{(2)}}\rangle\\
=&x_{\underline{(1)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes f^b\otimes e_{b\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{b\underline{(5)}}{}^{\overline{(\infty)}}\\
&\otimes x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)}k_{(4)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(2)}\otimes f^c\\
&\otimes e_{c\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} e_{b\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(3)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{b\underline{(4)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{c\underline{(3)}}{}^{\overline{(\infty)}}\otimes k_{(9)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(6)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}\otimes y_{(3)}\\
&\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(S(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(1)}), e_{b\underline{(5)}}{}^{\overline{(1)}})\\
&\mathcal{R}(S(k_{(8)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)}y_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}), e_{c\underline{(3)}}{}^{\overline{(1)}})\mathcal{R}(S(k_{(7)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)}), e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(4)}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(6)}k_{(6)})\\
&\mathcal{R}(e_{c\underline{(1)}}{}^{\overline{(1)}}, x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(5)}k_{(5)}e_{b\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}e_{b\underline{(4)}}{}^{\overline{(1)}}{}_{(3)})\langle y_{\underline{(1)}}{}^{\overline{(0)}}, e_{b\underline{(3)}}\rangle\langle y_{\underline{(2)}}{}^{\overline{(0)}}, e_{c\underline{(2)}}\rangle,
\end{aligned}
$$

using (2.2.4) to gather $e_{c\underline{(1)}}{}^{\overline{(1)}}$ and $e_{c\underline{(3)}}{}^{\overline{(1)}}$, and cancelling some $\mathcal{R}$s. In the final expression one can use (2.2.5) to change the order in the fifth tensor factor as well as inside $\mathcal{R}$, to recover our calculation of $(\mathrm{id}\otimes\Delta)\Delta(x\otimes k\otimes y)$ up to a change of basis labels. $\square$

**Lemma 3.2.6.** *The antipode of $B^{\underline{\mathrm{op}}}{>\!\!\!\triangleleft\kern-.33em\cdot\,} A {\cdot\kern-.33em\triangleright\!\!\!<} B^*$ in Theorem 3.2.2 is given by*

$$
\begin{aligned}
S(x\otimes k\otimes y)=&\overline{S}(e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}})\otimes S(x^{\overline{(1)}}{}_{(2)}k_{(2)}f^{a\overline{(1)}}{}_{(3)})\otimes \underline{S}f^{a\overline{(0)}}\\
&\mathcal{R}(f^{a\overline{(1)}}{}_{(1)}, S(x^{\overline{(1)}}{}_{(1)}k_{(1)}))\mathcal{R}(S^2(k_{(5)}y^{\overline{(1)}}{}_{(2)}), e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}x^{\overline{(1)}}{}_{(5)}e_{a\underline{(3)}}{}^{\overline{(1)}}{}_{(3)})\\
&\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, x^{\overline{(1)}}{}_{(3)}k_{(3)})\mathcal{R}(S(k_{(4)}y^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}}{}_{(1)})\langle y^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\\
&v(f^{a\overline{(1)}}{}_{(2)})u(e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}x^{\overline{(1)}}{}_{(4)}e_{a\underline{(3)}}{}^{\overline{(1)}}{}_{(2)}),
\end{aligned}
$$

*where $v(k)=\mathcal{R}(k_{(1)}, Sk_{(2)})$ and $u(k)=\mathcal{R}(S^2k_{(1)}, k_{(2)})$.*

*Proof.* We first compute $(S(x\otimes k\otimes y)_{(1)})(x\otimes k\otimes y)_{(2)}$, which on expanding out the product has in the first tensor factor

$$
\begin{aligned}
&\overline{S}(e_{b\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}})\cdot_{\underline{\mathrm{op}}} e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}}\\
=&(\underline{\epsilon}(e_{a\underline{(1)}})\underline{\epsilon}(e_{a\underline{(3)}})\underline{\epsilon}(x)\underline{\epsilon}(e_{b\underline{(1)}})\underline{\epsilon}(e_{b\underline{(3)}}))^{\overline{(\infty)}},
\end{aligned}
$$

which further collapses the full expression to give

$$
\begin{aligned}
(S(x\otimes k\otimes y)_{(1)})(x\otimes k\otimes y)_{(2)} = {}&\underline{\epsilon}(x)\otimes S(k_{(2)}f^{b\overline{(1)}}{}_{(4)})k_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}\otimes(\underline{S}f^{b\overline{(0)}})y_{\underline{(2)}}\\
&\langle f^a,e_b\rangle\langle y_{\underline{(1)}}{}^{\overline{(0)}},e_a\rangle\mathcal{R}(f^{b\overline{(1)}}{}_{(2)},Sk_{(1)})\mathcal{R}(f^{b\overline{(1)}}{}_{(1)},k_{(4)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})v(f^{b\overline{(1)}}{}_{(3)})\\
={}&\underline{\epsilon}(x)\otimes(Sy_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(5)})(Sk_{(2)})k_{(3)}y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}\otimes(\underline{S}y_{\underline{(1)}}{}^{\overline{(0)}\overline{(0)}})y_{\underline{(2)}}\\
&\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(3)},Sk_{(1)})\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(1)},y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(2)},k_{(4)})v(y_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(4)})\\
={}&\underline{\epsilon}(x)\otimes\epsilon(k)(Sy_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)}\otimes(\underline{S}y_{\underline{(1)}}{}^{\overline{(0)}})y_{\underline{(2)}}\mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(5)})v(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\\
={}&\underline{\epsilon}(x)\otimes\epsilon(k)\otimes(\underline{S}y_{\underline{(1)}}{}^{\overline{(0)}})y_{\underline{(2)}}\ \mathcal{R}(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})v(y_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\\
={}&\underline{\epsilon}(x)\otimes\epsilon(k)\otimes\underline{\epsilon}(y)=\epsilon(x\otimes k\otimes y).
\end{aligned}
$$

Similarly, on computing $(x\otimes k\otimes y)_{(1)}(S(x\otimes k\otimes y)_{(2)})$ we have $f^{a\overline{(0)}}\underline{S}f^{b\overline{(0)}}=(\underline{\epsilon}f^a\underline{\epsilon}f^b)^{\overline{(0)}}$ in the third tensor factor which collapses the expressions to give

$$
\begin{aligned}
(x\otimes k\otimes y)_{(1)}(S(x\otimes k\otimes y)_{(2)}) = {}&x_{\underline{(1)}}\cdot_{\underline{\mathrm{op}}}\overline{S}x_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(\infty)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\otimes\underline{\epsilon}y\\
&\mathcal{R}(S^2k_{(4)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)})\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}),x_{\underline{(2)}}{}^{\overline{(\infty)}\overline{(1)}})u(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\\
={}&x_{\underline{(1)}}\cdot_{\underline{\mathrm{op}}}\overline{S}x_{\underline{(2)}}{}^{\overline{(\infty)}}\otimes x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)}Sk_{(3)}Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}\otimes\underline{\epsilon}y\mathcal{R}(S^2k_{(4)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(6)})\\
&\mathcal{R}(Sk_{(1)}Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(6)})u(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\\
={}&x_{\underline{(1)}}\cdot_{\underline{\mathrm{op}}}\overline{S}x_{\underline{(2)}}{}^{\overline{(\infty)}}\otimes 1\otimes\underline{\epsilon}y\mathcal{R}(S^2k_{(2)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(Sk_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)})u(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\\
={}&x_{\underline{(1)}}\cdot_{\underline{\mathrm{op}}}\overline{S}x_{\underline{(2)}}{}^{\overline{(\infty)}}\otimes\epsilon k\otimes\underline{\epsilon}y\mathcal{R}(Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})u(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})=\epsilon(x\otimes k\otimes y).
\end{aligned}
$$

□

Finally, we show that the codouble bosonisation is coquasitriangular so as to have an inductive construction of such Hopf algebras.

**Proposition 3.2.7.** *The codouble bosonisation $coD_A(B)=B^{\underline{\mathrm{op}}}{\blacktriangleright\!\!\!<}A{\triangleright\!\!\!<}B^*$ is coquasitriangular with*

$$\mathcal{R}(x\otimes k\otimes y,w\otimes\ell\otimes z)=\langle\underline{S}z^{\overline{(0)}},x\rangle\mathcal{R}(k,\ell z^{\overline{(1)}})\underline{\epsilon}(y)\underline{\epsilon}(w)$$

*for all $x, w \otimes B^{\underline{\mathrm{op}}}$, $k, \ell \in A$, and $y, z \in B^*$.*

*Proof.* (i) Expanding the definitions of the product and the coquasitriangular structure,

$$
\begin{aligned}
&\mathcal{R}\Big((m \otimes j \otimes v), (x \otimes k \otimes y)(w \otimes \ell \otimes z)\Big)\\
&\quad =\langle \underline{S}(y^{\overline{(0)}\,\overline{(0)}} z^{\overline{(0)}}), m\rangle \mathcal{R}(j, k\ell_{(1)} y^{\overline{(0)}\,\overline{(1)}} z^{\overline{(1)}}) \mathcal{R}(y^{\overline{(1)}}, \ell_{(2)})\\
&\quad =\langle \underline{S}(y^{\overline{(0)}} z^{\overline{(0)}}), m\rangle \mathcal{R}(j, k\ell_{(1)} y^{\overline{(1)}}{}_{(1)} z^{\overline{(1)}}) \mathcal{R}(y^{\overline{(1)}}{}_{(2)}, \ell_{(2)})\\
&\quad =\langle \underline{S} z^{\overline{(0)}\,\overline{(0)}} \underline{S} y^{\overline{(0)}\,\overline{(0)}}, m\rangle \mathcal{R}(j, k\ell_{(1)} y^{\overline{(1)}}{}_{(1)} z^{\overline{(1)}}) \mathcal{R}(y^{\overline{(1)}}{}_{(2)}, \ell_{(2)}) \mathcal{R}(y^{\overline{(0)}\,\overline{(1)}}, z^{\overline{(0)}\,\overline{(1)}})\\
&\quad =\langle \underline{S} z^{\overline{(0)}}, m_{\underline{(1)}}\rangle \langle \underline{S} y^{\overline{(0)}}, m_{\underline{(2)}}\rangle \mathcal{R}(j, k\ell_{(1)} y^{\overline{(1)}}{}_{(2)} z^{\overline{(1)}}{}_{(2)}) \mathcal{R}(y^{\overline{(1)}}{}_{(3)}, \ell_{(2)}) \mathcal{R}(y^{\overline{(1)}}{}_{(1)}, z^{\overline{(1)}}{}_{(1)})\\
&\quad =\langle \underline{S} z^{\overline{(0)}}, m_{\underline{(1)}}\rangle \langle \underline{S} y^{\overline{(0)}}, m_{\underline{(2)}}\rangle \mathcal{R}(j, k y^{\overline{(1)}}{}_{(3)} \ell_{(2)} z^{\overline{(1)}}{}_{(2)}) \mathcal{R}(y^{\overline{(1)}}{}_{(1)}, z^{\overline{(1)}}{}_{(1)}) \mathcal{R}(y^{\overline{(1)}}{}_{(2)}, \ell_{(1)}).
\end{aligned}
$$

The second equality uses the right coaction on $y$. The third equality expands the braided-antipode $\underline{S}(y^{\overline{(0)}} z^{\overline{(0)}})$. The fourth equality uses the right-coaction on $y$ and $z$, and evaluation. The last equality uses quasicommutativity to change the order of product inside the first $\mathcal{R}$. On the other side,

$$
\begin{aligned}
&\mathcal{R}\Big((m \otimes j \otimes v)_{(1)}, w \otimes \ell \otimes z\Big) \mathcal{R}\Big((m \otimes j \otimes v)_{(2)}, x \otimes k \otimes y\Big)\\
&\quad =\mathcal{R}(m_{\underline{(1)}} \otimes m_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)} j_{(1)} \otimes f^a, w \otimes \ell \otimes z)\\
&\qquad \mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} m_{\underline{(2)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{a\underline{(3)}}{}^{\overline{(\infty)}} \otimes j_{(4)} v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)} \otimes v_{(2)}, x \otimes k \otimes y)\\
&\qquad \mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, m_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)} j_{(2)}) \mathcal{R}(S(j_{(3)} v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}), e_{a\underline{(3)}}{}^{\overline{(1)}}) \langle v_{\underline{(1)}}{}^{\overline{(0)}}, e_{a\underline{(2)}}\rangle\\
&\quad =\langle \underline{S} z^{\overline{(0)}}, m_{\underline{(1)}}\rangle \langle \underline{S} y^{\overline{(0)}}, m_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle \mathcal{R}(m_{\underline{(2)}}{}^{\overline{(1)}} j_{(1)}, \ell z^{\overline{(1)}}) \mathcal{R}(j_{(2)}, k y^{\overline{(1)}})\\
&\quad =\langle \underline{S} z^{\overline{(0)}}, m_{\underline{(1)}}\rangle \langle \underline{S} y^{\overline{(0)}\,\overline{(0)}}, m_{\underline{(2)}}\rangle \mathcal{R}(y^{\overline{(0)}\,\overline{(1)}} j_{(1)}, \ell z^{\overline{(1)}}) \mathcal{R}(j_{(2)}, k y^{\overline{(1)}})\\
&\quad =\langle \underline{S} z^{\overline{(0)}}, m_{\underline{(1)}}\rangle \langle \underline{S} y^{\overline{(0)}}, m_{\underline{(2)}}\rangle \mathcal{R}(y^{\overline{(1)}}{}_{(1)} j_{(1)}, \ell z^{\overline{(1)}}) \mathcal{R}(j_{(2)}, k y^{\overline{(1)}}{}_{(2)}).
\end{aligned}
$$

The third equality uses $\langle y^{\overline{(0)}}, m_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle m_{\underline{(2)}}{}^{\overline{(1)}} = \langle y^{\overline{(0)}\,\overline{(0)}}, m_{\underline{(2)}}\rangle y^{\overline{(0)}\,\overline{(1)}}$ and the fourth uses the right coaction on $y$. We can then use (2.2.4) to gather the parts of $j$ and obtain the same expression as on the first side. (ii) Similarly expanding the definitions,

$$
\begin{aligned}
&\mathcal{R}\Big((x{\otimes}k\otimes y)(w\otimes \ell\otimes z),(m\otimes j\otimes v)\Big)\\
&\quad=\langle \underline{S}v^{\overline{(0)}}, x\cdot_{\underline{\mathrm{op}}} w^{\overline{(\infty)}}\rangle \mathcal{R}(k_{(2)}\ell, jv^{\overline{(1)}})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}})\\
&\quad=\langle \underline{S}v^{\overline{(0)}}, w^{\overline{(\infty)}\,\overline{(\infty)}}x^{\overline{(\infty)}}\rangle \mathcal{R}(k_{(2)}\ell, jv^{\overline{(1)}})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}})\mathcal{R}(Sx^{\overline{(1)}}, w^{\overline{(\infty)}\,\overline{(1)}})\\
&\quad=\langle \underline{S}v^{\overline{(0)}}{}_{\underline{(1)}}{}^{\overline{(0)}}, x^{\overline{(\infty)}}\rangle\langle \underline{S}v^{\overline{(0)}}{}_{\underline{(2)}}{}^{\overline{(0)}}, w^{\overline{(\infty)}}\rangle \mathcal{R}(k_{(2)}\ell, jv^{\overline{(1)}})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}{}_{(1)})\\
&\qquad\mathcal{R}(Sx^{\overline{(1)}}, w^{\overline{(1)}}{}_{(2)})\mathcal{R}(v^{\overline{(0)}}{}_{\underline{(1)}}{}^{\overline{(1)}}, v^{\overline{(0)}}{}_{\underline{(2)}}{}^{\overline{(1)}})\\
&\quad=\langle \underline{S}v_{\underline{(1)}}{}^{\overline{(0)}}, x^{\overline{(\infty)}}\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}}, w^{\overline{(\infty)}}\rangle \mathcal{R}(k_{(2)}\ell, jv_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(Sk_{(1)}, w^{\overline{(1)}}{}_{(1)})\\
&\qquad\mathcal{R}(Sx^{\overline{(1)}}, w^{\overline{(1)}}{}_{(2)})\mathcal{R}(v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\\
&\quad=\langle \underline{S}v_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(0)}}, x\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}\,\overline{(0)}}, w\rangle \mathcal{R}(k_{(2)}\ell, jv_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(Sk_{(1)}, v_{\underline{(2)}}{}^{\overline{(0)}\,\overline{(1)}}{}_{(1)})\\
&\qquad\mathcal{R}(Sv_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(1)}}, v_{\underline{(2)}}{}^{\overline{(0)}\,\overline{(1)}}{}_{(2)})\mathcal{R}(v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\\
&\quad=\langle \underline{S}v_{\underline{(1)}}{}^{\overline{(0)}}, x\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}}, w\rangle \mathcal{R}(k_{(2)}\ell, jv_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(Sk_{(1)}, v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\\
&\qquad\mathcal{R}(Sv_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}, v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}, v_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\\
&\quad=\langle \underline{S}v_{\underline{(1)}}{}^{\overline{(0)}}, x\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}}, w\rangle \mathcal{R}(k, j_{(1)}v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(\ell, j_{(2)}v_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}v_{\underline{(2)}}{}^{\overline{(1)}}).
\end{aligned}
$$

The second equality expands the braided-product $\cdot_{\underline{\mathrm{op}}}$. The third equality uses the left-coaction on $w$, followed by the duality pairing and taking $\underline{S}$ to the left in $\underline{\Delta}(\underline{S}v^{\overline{(0)}})$. The fourth equality uses the comodule coalgebra property (2.1.7) on $v$ and the right coaction axioms. The fifth equality moves the coactions onto $x, w$ by duality. The sixth equality is similar to the fourth. For the last equality we cancel the last two $\mathcal{R}$s and use (2.2.4) to gather $k$ inside $\mathcal{R}$ and cancel further. On the other side,

$$
\begin{aligned}
&\mathcal{R}\Big((x\otimes k\otimes y),(m\otimes j\otimes v)_{(1)}\Big)\mathcal{R}\Big((w\otimes \ell\otimes z),(m\otimes j\otimes v)_{(2)}\Big)\\
&\quad=\langle \underline{S}f^{a\overline{(0)}}, x\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}}, w\rangle\langle v_{\underline{(1)}}{}^{\overline{(0)}}, e_a\rangle \mathcal{R}(k, j_{(1)}f^{a\overline{(1)}})\mathcal{R}(\ell, j_{(2)}v_{\underline{(1)}}{}^{\overline{(1)}}v_{\underline{(2)}}{}^{\overline{(1)}})\\
&\quad=\langle \underline{S}v_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(0)}}, x\rangle\langle \underline{S}v_{\underline{(2)}}{}^{\overline{(0)}}, w\rangle \mathcal{R}(k, j_{(1)}v_{\underline{(1)}}{}^{\overline{(0)}\,\overline{(1)}})\mathcal{R}(\ell, j_{(2)}v_{\underline{(1)}}{}^{\overline{(1)}}v_{\underline{(2)}}{}^{\overline{(1)}})
\end{aligned}
$$

on substituting $f^a = v_{\underline{(1)}}{}^{\overline{(0)}}$. We can then use the right coaction property on $v_{\underline{(1)}}$ to recover the result of our first calculation. (iii) We expand the definitions to compute

$$
\begin{aligned}
&(x\otimes k\otimes y)_{(2)}(w\otimes \ell\otimes z)_{(2)}\mathcal{R}\Big((x\otimes k\otimes y)_{(1)},(w\otimes \ell\otimes z)_{(1)}\Big)\\
&=x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}e_{b\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\mathrm{op}}w^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{b\underline{(3)}}{}^{\overline{(\infty)}}\otimes k_{(3)}\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(Sk_{(2)},e_{b\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}w^{\overline{(1)}}{}_{(3)}e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(y^{\overline{(1)}},\ell_{(5)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(e_{b\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},w^{\overline{(1)}}{}_{(2)}\ell_{(2)})\\
&\quad\mathcal{R}(S(\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),e_{b\underline{(3)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(x_{\underline{(2)}}{}^{\overline{(1)}}k_{(1)},w^{\overline{(1)}}{}_{(1)}\ell_{(1)}f^{b\overline{(1)}})\langle\underline{S}f^{b\overline{(0)}},x_{\underline{(1)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}},e_{b\underline{(2)}}\rangle\\
&=x_{\underline{(4)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}^{-1}x_{\underline{(3)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}w^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}\overline{S}^{-1}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes k_{\underline{(3)}}\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(Sk_{(2)},x_{(3)}{}^{\overline{(1)}}{}_{(5)}w^{\overline{(1)}}{}_{(3)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(y^{\overline{(1)}},\ell_{(5)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(4)},w^{\overline{(1)}}{}_{(2)}\ell_{(2)})\\
&\quad\mathcal{R}(S(\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(x_{\underline{(4)}}{}^{\overline{(1)}}k_{(1)},w^{\overline{(1)}}{}_{(1)}\ell_{(1)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)})\\
&\quad\mathcal{R}(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(2)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(x_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\langle z_{\underline{(1)}}{}^{\overline{(0)}},\overline{S}^{-1}x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle\\
&=w^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes k_{(3)}\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(Sk_{(2)},w^{\overline{(1)}}{}_{(2)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(y^{\overline{(1)}},\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S(\ell_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\\
&\quad\mathcal{R}(k_{(1)},w^{\overline{(1)}}{}_{(1)}\ell_{(1)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\langle z_{\underline{(1)}}{}^{\overline{(0)}},\overline{S}^{-1}x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle\\
&=w^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes k_{(3)}\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\\
&\quad\mathcal{R}(Sk_{(2)},w^{\overline{(1)}}{}_{(2)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(y^{\overline{(1)}},\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S(\ell_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}),x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\\
&\quad\mathcal{R}(k_{(1)},w^{\overline{(1)}}{}_{(1)}\ell_{(1)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}z_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(1)})\mathcal{R}(z_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}{}_{(2)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}\rangle\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes k_{(3)}\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}\rangle\\
&\quad\mathcal{R}(Sk_{(2)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(y^{\overline{(1)}},\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S\ell_{(2)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(k_{(1)},\ell_{(1)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)})\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes k_{(2)}\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\ \langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}\rangle\\
&\quad\mathcal{R}(y^{\overline{(1)}},\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(k_{(1)},\ell_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)})\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes \ell_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}\rangle\\
&\quad\mathcal{R}(y^{\overline{(1)}},\ell_{(4)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(k_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}).
\end{aligned}
$$

The second equality uses the duality $\langle\underline{S}f^{b\overline{(0)}},x_{\underline{(1)}}\rangle f^{b\overline{(1)}}=\langle f^b,\overline{S}^{-1}x_{\underline{(1)}}{}^{\overline{(\infty)}}\rangle x_{\underline{(1)}}{}^{\overline{(1)}}$ to substitute $e_b=\overline{S}^{-1}x_{\underline{(1)}}{}^{\overline{(\infty)}}$ in all the places where it occurs, and the comodule algebra property (2.1.8). The third equality cancels $(x_{\underline{(4)}}\cdot_{\underline{\mathrm{op}}}\overline{S}^{-1}x_{\underline{(3)}})^{\overline{(\infty)}}$ resulting in trivial coactions. We use the duality $\langle z_{\underline{(1)}}{}^{\overline{(0)}},\overline{S}^{-1}x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle x_{\underline{(2)}}{}^{\overline{(1)}}=\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}\rangle z_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}$ for the fourth equality and gather $w^{\overline{(1)}}$ inside $\mathcal{R}$ to cancel it for the fifth one. The sixth equality uses (2.2.5) in

$$\ell_{(1)}x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}\overline{\mathcal{R}}(x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)},\ell_{(2)}) = x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}\ell_{(2)}\overline{\mathcal{R}}(x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},\ell_{(2)})$$

and then gathers the parts of $x_{\underline{(1)}}{}^{\overline{(1)}}$, and cancels some $\mathcal{R}$s. We finally use (2.2.5) to change the order of products in the third tensor factor. On the other side,

$$\begin{aligned}
&(w\otimes\ell\otimes z)_{(1)}(x\otimes k\otimes y)_{(1)}\mathcal{R}\Big((x\otimes k\otimes y)_{(2)},(w\otimes\ell\otimes z)_{(2)}\Big)\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes\ell_{(2)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes f^{b\overline{(0)}}f^a\langle\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}},e_{a\underline{(1)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}x_{\underline{(2)}}{}^{\overline{(\infty)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}}\rangle\\
&\quad\mathcal{R}(k_{(5)}y^{\overline{(1)}}{}_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}z_{\underline{(2)}}{}^{\overline{(1)}})\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(f^{b\overline{(1)}},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\\
&\quad\mathcal{R}(S(k_{(4)}y^{\overline{(1)}}{}_{(1)}),e_{a\underline{(3)}}{}^{\overline{(1)}})\ \langle y^{\overline{(0)}},e_{a\underline{(2)}}\rangle\langle z_{\underline{(1)}}{}^{\overline{(0)}},e_b\rangle\\
&=w\cdot_{\mathrm{op}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes\ell_{(2)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes z_{\underline{(1)}}{}^{\overline{(0)}}f^a\langle\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}},(\underline{S}^{-1}e_{a\underline{(3)}}{}^{\overline{(\infty)}})x_{\underline{(2)}}{}^{\overline{(\infty)}}e_{a\underline{(1)}}{}^{\overline{(\infty)}}\rangle\\
&\quad\mathcal{R}(k_{(5)}y^{\overline{(1)}}{}_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(2)}}{}^{\overline{(1)}})\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\\
&\quad\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\mathcal{R}(S(k_{(4)}y^{\overline{(1)}}{}_{(1)}),e_{a\underline{(3)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(Se_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(2)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\\
&\quad\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}e_{a\underline{(1)}}{}^{\overline{(1)}}{}_{(3)}),e_{a\underline{(3)}}{}^{\overline{(1)}}{}_{(2)})\langle y^{\overline{(0)}},e_{a\underline{(2)}}\rangle\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes\ell_{(2)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes z_{\underline{(1)}}{}^{\overline{(0)}}f^a\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)}k_{(2)})\\
&\quad\mathcal{R}(z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}k_{(3)})\mathcal{R}(S(k_{(4)}y^{\overline{(1)}}{}_{(1)}),z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)},x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\\
&\quad\mathcal{R}(S(x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)}),z_{\underline{(4)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(k_{(5)}y^{\overline{(1)}}{}_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(5)}z_{\underline{(3)}}{}^{\overline{(1)}}{}_{(3)}z_{\underline{(4)}}{}^{\overline{(1)}}{}_{(5)})\\
&\quad\mathcal{R}(z_{\underline{(3)}}{}^{\overline{(1)}}{}_{(1)},z_{\underline{(4)}}{}^{\overline{(1)}}{}_{(3)})\mathcal{R}(z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)},z_{\underline{(4)}}{}^{\overline{(1)}}{}_{(4)})\langle\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}},e_{a\underline{(1)}}{}^{\overline{(\infty)}}\rangle\langle y^{\overline{(0)}},e_{a\underline{(2)}}\rangle\langle\underline{S}z_{\underline{(4)}}{}^{\overline{(0)}},\underline{S}^{-1}e_{a\underline{(3)}}{}^{\overline{(\infty)}}\rangle\\
&\quad\langle\underline{S}z_{\underline{(3)}}{}^{\overline{(0)}},x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes\ell_{(2)}x_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}{}^{\overline{(0)}}\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}})\mathcal{R}(S(k_{(2)}y^{\overline{(1)}}{}_{(1)}),z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\\
&\quad\mathcal{R}(Sx_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)},z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(k_{(3)}y^{\overline{(1)}}{}_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)}z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)},z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle\\
&=w\cdot_{\underline{\mathrm{op}}}x_{\underline{(1)}}{}^{\overline{(\infty)}}\otimes\ell_{(2)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)}k_{(1)}\otimes y^{\overline{(0)}}z_{\underline{(2)}}{}^{\overline{(0)}}\mathcal{R}(S\ell_{(1)},x_{\underline{(1)}}{}^{\overline{(1)}}{}_{(1)})\mathcal{R}(S(k_{(2)}y^{\overline{(1)}}{}_{(1)}),z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(1)})\\
&\quad\mathcal{R}(k_{(3)}y^{\overline{(1)}}{}_{(2)},\ell_{(3)}z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(4)}z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(4)})\mathcal{R}(Sz_{\underline{(1)}}{}^{\overline{(1)}}{}_{(2)},z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(2)})\mathcal{R}(z_{\underline{(1)}}{}^{\overline{(1)}}{}_{(3)},z_{\underline{(2)}}{}^{\overline{(1)}}{}_{(3)})\langle\underline{S}z_{\underline{(1)}}{}^{\overline{(0)}},x_{\underline{(1)}}{}^{\overline{(1)}}\rangle.
\end{aligned}$$

The second equality uses $\langle z_{\underline{(1)}}{}^{\overline{(0)}},e_b\rangle$ to substitute $f^b=z_{\underline{(1)}}{}^{\overline{(0)}}$ and we then expand $\cdot_{\underline{\mathrm{op}}}$

inside the pairing. For the third equality, we use

$$\begin{aligned}&\langle \underline{S}z_{\underline{(2)}}{}^{\overline{(0)}}, (\underline{S}^{-1}e_{a\underline{(3)}}{}^{\overline{(\infty)}})x_{\underline{(2)}}{}^{\overline{(\infty)}}e_{a\underline{(1)}}{}^{\overline{(\infty)}}\rangle \\ &\quad =\langle (\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})_{\underline{(1)}}, \underline{S}^{-1}e_{a\underline{(3)}}{}^{\overline{(\infty)}}\rangle\langle (\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})_{\underline{(2)}}, x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle\langle (\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})_{\underline{(3)}}, e_{a\underline{(1)}}{}^{\overline{(\infty)}}\rangle\end{aligned}$$

and move $\underline{S}$ to the left in $\underline{\Delta}^2(\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})$. For the fourth equality we gather the coproducts of $e_a$ to give $\langle(\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})y^{\overline{(0)}}z_{\underline{(4)}}{}^{\overline{(0)}}, e_a\rangle$ so that we can set $f^a = (\underline{S}z_{\underline{(2)}}{}^{\overline{(0)}})y^{\overline{(0)}}z_{\underline{(4)}}{}^{\overline{(0)}}$, allowing us to cancel $(z_{\underline{(1)}}\underline{S}z_{\underline{(2)}})^{\overline{(0)}}$ and drop out following coactions. For the fifth equality, we use the duality pairing $\langle \underline{S}z_{\underline{(1)}}{}^{\overline{(0)}}, x_{\underline{(2)}}{}^{\overline{(\infty)}}\rangle x_{\underline{(2)}}{}^{\overline{(1)}} = \langle \underline{S}z_{\underline{(1)}}{}^{\overline{(0)}\overline{(0)}}, x_{\underline{(2)}}\rangle z_{\underline{(1)}}{}^{\overline{(0)}\overline{(1)}}$ and then gather $z_{\underline{(2)}}{}^{\overline{(1)}}$ inside $\mathcal{R}$ so as to cancel it and recover the result of our first calculation. □

## 3.3 Dual Basis of $\mathfrak{c}_q[SL_2]$ by Codouble Bosonisation

The coquasitriangular Hopf algebra $\mathbb{C}_q[SL_2]$ in some standard conventions is generated by $a, b, c, d$ with the relations,

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc,$$

$$ad - q^{-1}bc = 1, \quad da - ad = (q - q^{-1})bc,$$

a 'matrix' form of coproduct (so $\Delta a = a\otimes a + b\otimes c$ etc.), $\epsilon(a) = \epsilon(d) = 1$, $\epsilon(b) = \epsilon(c) = 0$ and antipode $Sa = d, Sd = a, Sb = -qb, Sc = -q^{-1}c$. The reduced version $c_q[SL_2]$ has

$$a^n = 1 = d^n, \quad b^n = 0 = c^n$$

as additional relations when $q$ is a primitive $n$-th root of unity. We will show how some version of this is obtained by codouble bosonisation.

Let $A = \mathbb{C}_q[t]/(t^n-1)$ be a coquasitriangular Hopf algebra with $t$ grouplike and $\mathcal{R}(t^r, t^s) = q^{rs}$. Also let $B = \mathbb{C}[x]/(x^n)$ be a braided Hopf algebra in ${}^A\mathcal{M}$ with

$$\Delta_L x = t\otimes x, \quad \underline{\Delta}x = 1\otimes x + x\otimes 1, \quad \underline{\epsilon}x = 0, \quad \underline{S}x = -x, \quad \Psi(x^r\otimes x^s) = q^{rs}x^s\otimes x^r.$$

The dual $B^* = \mathbb{C}[y]/(y^n)$ lives in $\mathcal{M}^A$ with the same form of coproduct, etc., as for $B$, but with right-coaction $\Delta_R y = y \otimes t$. We choose pairing $\langle x, y\rangle = 1$ and take a basis of $B$ and a dual basis of $B^*$ respectively as

$$\{e_a\} = \{x^a\}_{0\le a<n}, \quad \{f^a\} = \left\{\frac{y^a}{[a]_q!}\right\}_{0\le a<n},$$

where $[a]_q$ is a $q$-integer defined by $[a]_q = (1-q^a)/(1-q)$ and $[a]_q! = [a]_q[a-1]_q\cdots[1]_q!$ with $[0]_q! = 1$. We also write $\left[\begin{smallmatrix} a \\ r \end{smallmatrix}\right]_q = \frac{[a]_q!}{[r]_q![a-r]_q!}$. We write $x^{a(\underline{\mathrm{op}})} = x \cdot_{\underline{\mathrm{op}}} x \cdot_{\underline{\mathrm{op}}} \cdots \cdot_{\underline{\mathrm{op}}} x$ with $a$-many $x$, and find inductively that

$$x^a = q^{\frac{a(a-1)}{2}} x^{a(\underline{\mathrm{op}})}, \quad \overline{S}(x^a) = (-1)^a x^{a(\underline{\mathrm{op}})}. \tag{3.3.1}$$

**Theorem 3.3.1.** *Let $q$ be a primitive $n$-th root of unity and $A, B, B^*$ be as above.*

1. *The codouble bosonisation of $B$, denoted $\mathfrak{c}_q[SL_2]$, has generators $x, t, y$ and*

$$\begin{aligned}
&x^n = y^n = 0, \quad t^n = 1, \quad yx = xy, \quad xt = qtx, \quad yt = qty,\\
&\Delta t = q\sum_{a=0}^{n-2}(q-1)^{a-1}(1-q^{-a-1})ty^a \otimes x^a t\\
&\quad = \frac{t}{q-1}\left(q\frac{1-((q-1)y\otimes x)^{n-1}}{1-(q-1)y\otimes x} - \frac{1-((1-q^{-1})y\otimes x)^{n-1}}{1-(1-q^{-1})y\otimes x}\right)t,\\
&\Delta x = x\otimes 1 + \sum_{a=0}^{n-2}(q-1)^a ty^a \otimes x^{a+1} = x\otimes 1 + t\left(\frac{1-((q-1)y\otimes x)^{n-1}}{1-(q-1)y\otimes x}\right)x,\\
&\Delta y = 1\otimes y + \sum_{a=0}^{n-2}(1-q^{-1})^a y^{a+1}\otimes x^a t = 1\otimes y + y\left(\frac{1-((1-q^{-1})y\otimes x)^{n-1}}{1-(1-q^{-1})y\otimes x}\right)t.
\end{aligned}$$

2. *If $n = 2m+1$, there is an isomorphism $\phi : \mathfrak{c}_q[SL_2] \to c_{q^{-m}}[SL_2]$ defined by*

$$\phi(x) = bd^{-1}, \quad \phi(t) = d^{-2}, \quad \phi(y) = \frac{d^{-1}c}{q^m - q^{-m}}.$$

*Proof.* (1) First we determine the products

$$(1\otimes 1\otimes y)(x\otimes 1\otimes 1) = x^{\overline{(\infty)}}\otimes 1\otimes y^{\overline{(0)}}\ \mathcal{R}(y^{\overline{(1)}},1)\mathcal{R}(S1,x^{\overline{(1)}}) = x\otimes 1\otimes y,$$

$$(1\otimes t\otimes 1)(x\otimes 1\otimes 1) = x^{\overline{(\infty)}}\otimes t\otimes 1\ \mathcal{R}(St,x^{\overline{(1)}}) = q^{-1}x\otimes t\otimes 1,$$

$$(1\otimes 1\otimes y)(1\otimes t\otimes 1) = 1\otimes t\otimes y^{\overline{(0)}}\ \mathcal{R}(y^{\overline{(1)}},t) = q1\otimes t\otimes y$$

as stated. The algebra generated by $x, y, t$ with these relations is $n^3$ dimensional, hence these are all the relations we need. Before go further, we note the $q$-identities

$$\sum_{r=0}^{a}(-1)^r\frac{q^{\frac{r(r+1)}{2}}}{[r]_q![a-r]_q!} = (1-q)^a,\ \sum_{r=0}^{a}q^r(-1)^r\frac{q^{\frac{r(r+1)}{2}}}{[r]_q![a-r]_q!} = (1-q)^a[a+1]_q. \quad (3.3.2)$$

Then, using (3.3.1), we compute

$$\begin{aligned}\Delta(1\otimes t\otimes 1) &= \sum_{a=0}^{n-1}\sum_{r=0}^{a}\begin{bmatrix}a\\r\end{bmatrix}_q 1\otimes t\otimes\frac{y^a}{[a]_q!}\otimes x^r\cdot_{\underline{\mathrm{op}}}\overline{S}x^{a-r}\otimes t\otimes 1\mathcal{R}(t^r,t)\mathcal{R}(St,t^{a-r})\\ &=\sum_{a=0}^{n-1}\sum_{r=0}^{a}\begin{bmatrix}a\\r\end{bmatrix}_q(-1)^{a-r}q^{\frac{r(r-1)}{2}+2r-a}1\otimes t\otimes\frac{y^a}{[a]_q!}\otimes x^{r(\underline{\mathrm{op}})}\cdot_{\underline{\mathrm{op}}}x^{(a-r)(\underline{\mathrm{op}})}\otimes t\otimes 1\\ &=\sum_{a=0}^{n-2}\sum_{r=0}^{a}\frac{(-1)^{a-r}\left[\begin{smallmatrix}a\\r\end{smallmatrix}\right]_q q^{\frac{r(r+1)}{2}}q^{-a+r}}{[a]_q!}ty^a\otimes x^a t\end{aligned}$$

since there is no contribution when $a = n-1$. We then use (3.3.2). Similarly,

$$\begin{aligned}\Delta(x\otimes 1\otimes 1) &= x\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\\ &\quad+\sum_{a=0}^{n-1}\sum_{r=0}^{a}\begin{bmatrix}a\\r\end{bmatrix}_q 1\otimes t\otimes\frac{y^a}{[a]_q!}\otimes x^r\cdot_{\underline{\mathrm{op}}}x\cdot_{\underline{\mathrm{op}}}\overline{S}x^{a-r}\otimes 1\otimes 1\mathcal{R}(t^r,t\\ &= x\otimes 1\otimes 1\otimes 1\otimes 1\otimes 1\\ &\quad+\sum_{a=0}^{n-1}\sum_{r=0}^{a}\begin{bmatrix}a\\r\end{bmatrix}_q q^{\frac{r(r+1)}{2}}1\otimes t\otimes\frac{y^a}{[a]_q!}\otimes x^{r(\underline{\mathrm{op}})}\cdot_{\underline{\mathrm{op}}}x\cdot_{\underline{\mathrm{op}}}(-1)^{a-r}x^{(a-r)(\underline{\mathrm{op}})}\otimes 1\otimes 1\\ &= x\otimes 1+\sum_{a=0}^{n-2}\sum_{r=0}^{a}\frac{(-1)^{a-r}\left[\begin{smallmatrix}a\\r\end{smallmatrix}\right]_q q^{\frac{r(r+1)}{2}}}{[a]_q!}ty^a\otimes x^{a+1},\end{aligned}$$

where for $a = n-1$, we will have the term $ty^{n-1}\otimes x^n = 0$. We again use (3.3.2). Finally, we use $\underline{\Delta}^2(e_a)=\underline{\Delta}^2(x^a)=\sum_{r=0}^{a}\sum_{s=0}^{r}\left[\begin{smallmatrix}a\\ r\end{smallmatrix}\right]_q\left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right]_q x^s\otimes x^{r-s}\otimes x^{a-r}$ to find

$$\begin{aligned}
&\Delta(1\otimes 1\otimes y)=1\otimes 1\otimes 1\otimes 1\otimes 1\otimes y\\
&\quad+\sum_{a=0}^{n-1}1\otimes 1\otimes f^a\otimes e_{a\underline{(1)}}\cdot_{\underline{\mathrm{op}}}\overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}}\otimes t\otimes 1\mathcal{R}(St,e_{a\underline{(3)}}{}^{\overline{(1)}})\langle y,e_{a\underline{(2)}}\rangle.\\
&=1\otimes 1\otimes 1\otimes 1\otimes 1\otimes y\\
&\quad+\sum_{a=0}^{n-1}\sum_{r=0}^{a}\sum_{s=0}^{r}\begin{bmatrix}a\\ r\end{bmatrix}_q\begin{bmatrix}r\\ s\end{bmatrix}_q 1\otimes 1\otimes\frac{y^a}{[a]_q!}\otimes x^s\cdot_{\underline{\mathrm{op}}}\overline{S}x^{a-r}\otimes t\otimes 1\ \mathcal{R}(St,t^{a-r})\langle y,x^{r-s}\rangle\\
&=1\otimes 1\otimes 1\otimes 1\otimes 1\otimes y\\
&\quad+\sum_{a=0}^{n-1}\sum_{r=0}^{a}\sum_{s=0}^{r}\begin{bmatrix}a\\ r\end{bmatrix}_q\begin{bmatrix}r\\ s\end{bmatrix}_q\delta_{1,r-s}q^{\frac{s(s-1)}{2}}(-q^{-1})^{-a+r}1\otimes 1\otimes\frac{y^a}{[a]_q!}\otimes x^{s\underline{(\mathrm{op})}}\cdot_{\underline{\mathrm{op}}}x^{(a-r)\underline{(\mathrm{op})}}\otimes t\otimes 1\\
&=1\otimes y+\sum_{a=0}^{n-1}\sum_{r=0}^{a}\sum_{s=0}^{r}\frac{(-1)^{a-r}\delta_{1,r-s}\left[\begin{smallmatrix}a\\ r\end{smallmatrix}\right]_q\left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right]_q q^{\frac{s(s-1)}{2}}q^{-a+r}}{[a]_q!}y^a\otimes x^{a-r+s}t\\
&=1\otimes y+\sum_{a=0}^{n-2}\sum_{r=0}^{a}\frac{(-1)^{a-r}\left[\begin{smallmatrix}a\\ r\end{smallmatrix}\right]_q q^{\frac{r(r+1)}{2}}q^{-a}}{[a]_q!}y^{a+1}\otimes x^a t.
\end{aligned}$$

There was no contribution from $a=0$ and for $a>0$ we needed $s=r-1$ for a contribution. We then use (3.3.2). Theorem 3.2.2 ensures that the Hopf algebra is coquasitriangular with $\mathcal{R}(t,t)=q$, $\mathcal{R}(x,y)=-1$, and zero otherwise.

(2) If $n=2m+1$ then $\varphi: c_{q^{-m}}[SL_2]\to\mathfrak{c}_q[SL_2]$ defined by

$$\varphi(a)=t^{m+1}+(q^m-q^{-m})xt^my,\quad \varphi(b)=xt^m,\quad \varphi(c)=(q^m-q^{-m})t^my,\quad \varphi(d)=t^m.$$

is an algebra map and inverse to $\phi$. Tedious but straightforward calculation gives

$$\Delta(\varphi(d))=\Delta t^m=t^m\otimes t^m+(q^{2m}-1)t^my\otimes t^mx=t^m\otimes t^m+(q^m-q^{-m})t^my\otimes xt^m,$$

to prove that $\Delta(\varphi(d))=(\varphi\otimes\varphi)\Delta d$. The coalgebra map property on the other generators then follows using this formula for $\Delta t^m$. Furthermore, the coquasitriangular structure from Lemma 3.2.7 computed on $\varphi(a), \varphi(b), \varphi(c), \varphi(d)$ as a matrix $\varphi(t^i{}_j)$ is

$$R^I{}_J = q^{m(m+1)} \begin{pmatrix} q^{-m} & 0 & 0 & 0 \\ 0 & 1 & q^{-m} - q^m & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-m} \end{pmatrix}. \tag{3.3.3}$$

for the values of $\mathcal{R}(\varphi(t^i{}_j), \varphi(t^k{}_l))$ where $I = (i, k)$ is (1,1),(1,2),(2,1), or (2,2) and similarly for $J = (j, l)$. If we set $p = q^{-m}$ then any power of $p$ is also a $2m+1$-th root of unity and $q = q^{-2m} = p^2$ so that our Hopf algebra is $c_p[SL_2]$ with its standard coquasitriangular structure with the correct factor $q^{m(m+1)} = p^{-m-1} = p^m = p^{-\frac{1}{2}}$. □

We now recall explicitly that for $q$ a primitive $n$-th root of unity and $q^2 \neq 1$, $u_q(sl_2)$ is generated by $E, F, K$, with relations, coproducts and coquasitriangular structure

$$E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-2}E, \quad KFK^{-1} = q^2F, \quad [E, F] = K - K^{-1},$$

$$\Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E,$$

$$\mathcal{R} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-2ab}}{[r]_{q^{-2}}!} F^r K^a \otimes E^r K^b,$$

where in our conventions we do not divide by the usual $q - q^{-1}$ in the $[E, F]$-relation (and where we use $q^{-2}$ rather than $q^2$ in the remaining relations compared with [20]). One can consider this as an unconventional normalisation of $E$ which is cleaner when we are not interested in a classical limit. It gives a commutative Hopf algebra $u_{-1}(sl_2)$ when $q = -1$. We first show that double bosonisation gives us some version of such reduced quantum groups, agreeing for primitive odd roots. This was outlined in [20, Example 17.6] in the odd root case but we give a short derivation for all roots.

**Lemma 3.3.2.** *[20] Let $q$ be a primitive $n$-th root of unity and let $H = \mathbb{C}_q\mathbb{Z}_n = \mathbb{C}_q[K]/(K^n - 1)$ be a quasitriangular Hopf algebra by $\mathcal{R}_K = \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} K^a \otimes K^b$ as in [19]. Let $B = \mathbb{C}[E]/(E^n)$ be a braided Hopf algebra in $\mathcal{M}_H$ and dual $B^* = \mathbb{C}[F]/(F^n)$*

*in ${}_H\mathcal{M}$ with actions $E \triangleleft K = qE$ and $K \triangleright F = qF$.*

1. *The double bosonisation $B^{*\underline{\mathrm{cop}}}{>\!\!\!\triangleleft}H{\triangleright\!\!\!<}B$ is a quasitriangular Hopf algebra, which we denote $\mathfrak{u}_q(sl_2)$, with the same coalgebra structure as above but with*

$$E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-1}E, \quad KFK^{-1} = qF, \quad [E,F] = K - K^{-1},$$

$$\mathcal{R}_{\mathfrak{u}_q(sl_2)} = \frac{1}{n}\sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-ab}}{[r]_{q^{-1}}!} F^r K^a \otimes E^r K^b.$$

2. *If $n = 2m+1$ then $\mathfrak{u}_q(sl_2)$ is isomorphic to $u_{q^{-m}}(sl_2)$ with its standard quasitriangular structure.*

*Proof.* Here $EK \equiv (1 \otimes E)(K \otimes 1) = K \otimes E \triangleleft K = K \otimes qE \equiv qKE$ and $KF \equiv (1 \otimes K)(F \otimes 1) = K \triangleright F \otimes K = qF \otimes K \equiv qFK$. From the cross relations stated in Theorem 3.1.1, we also have

$$\begin{aligned} EF =& FE + \frac{1}{n}\sum_{a,b=0}^{n-1} q^{-ab}K^b\langle F, E \triangleleft K^a\rangle - \frac{1}{n}\sum_{a,b=0}^{n-1} q^{ab}K^a\langle K^b \triangleright F, E\rangle \\ =& FE + \frac{1}{n}\sum_{b=0}^{n-1}\left(\frac{1-q^{-n(b-1)}}{1-q^{-(b-1)}}\right)K^b\langle F, E\rangle - \frac{1}{n}\sum_{a=0}^{n-1}\left(\frac{1-q^{n(a+1)}}{1-q^{a+1}}\right)K^a\langle F, E\rangle \\ =& FE + K - K^{-1}, \end{aligned}$$

where we choose $\langle F, E\rangle = 1$. This is the same choice of normalisation for the braided line duality as in the calculation in Theorem 3.3.1. For the coproduct, clearly $\Delta K = K \otimes K$ while $\Delta E \equiv \Delta(1 \otimes E) = 1 \otimes 1 \otimes 1 \otimes E + 1 \otimes E \triangleleft \mathcal{R}_K^{(1)} \otimes \mathcal{R}_K^{(2)} \otimes 1 = 1 \otimes 1 \otimes 1 \otimes E + 1 \otimes E \otimes K \otimes 1 \equiv 1 \otimes E + E \otimes K$ and $\Delta F \equiv \Delta(F \otimes 1) = F \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \mathcal{R}_K^{-(1)} \otimes \mathcal{R}_K^{-(2)} \triangleright F \otimes 1 = F \otimes 1 \otimes 1 \otimes 1 + 1 \otimes K^{-1} \otimes F \otimes 1 \equiv F \otimes 1 + K^{-1} \otimes F$. Hence we have the relations and coalgebra as stated. Also from Theorem 3.1.1,

$$\mathcal{R}_{\mathfrak{u}_q(sl_2)} = \sum_{r=0}^{n-1}\left(\frac{F^r}{[r]_q!} \otimes \underline{S}E^r\right)\mathcal{R}_K = \frac{1}{n}\sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{\frac{r(r-1)}{2}} q^{-ab}}{[r]_q!} F^r K^a \otimes E^r K^b,$$

which we write as stated. When $n = 2m + 1$, it is easy to see that the relations and quasitriangular structure become those of $u_p(sl_2)$ with $p = q^{-m}$, which are the same as in [20] after allowing for the normalisation of the generators. Note that if $q$ is an even root of unity then $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ need not be factorisable, see Example 3.3.4. In fact, $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ is factorisable iff $n$ is odd, which can be proven in a similar way to the proof in [17]. $\square$

We see that the double bosonisation $\mathfrak{u}_q(sl_2)$ recovers $u_p(sl_2)$ in the odd root of unity case with $p = q^{\frac{1}{2}}$, in line with the generic $q$ case in [28]. Clearly $\mathfrak{u}_q(sl_2)$ has a PBW-type basis $\{F^iK^jE^k\}_{0\le i,j,k\le n-1}$ as familiar in the odd case for $u_p(sl_2)$.

**Corollary 3.3.3.** *The basis $\{x^it^jy^k\}_{0\le i,j,k\le n-1}$ of $\mathfrak{c}_q[SL_2]$ is, up to normalisation, dual to the PBW basis of $\mathfrak{u}_q(sl_2)$ in the sense*

$$\langle x^it^jy^k, F^{i'}K^{j'}E^{k'}\rangle = \delta_{i,i'}\delta_{k,k'}q^{jj'}[i]_{q^{-1}}![k]_q!.$$

*More precisely,* $\left\{\dfrac{x^i\delta_j(t)y^k}{[i]_{q^{-1}}![k]_q!}\right\}_{0\le i,j,k<n}$ *is a dual basis to* $\{F^iK^jE^k\}_{0\le i,j,k<n}$*, where* $\delta_j(t) = \frac{1}{n}\sum\limits_{l=0}^{n-1} q^{-jl}t^l$*.*

*Proof.* The duality pairing between the double and codouble bosonisations is

$$\langle x^it^jy^k, F^{i'}K^{j'}E^{n'}\rangle = \langle x^{i(\underline{\mathrm{op}})}, F^{i'}\rangle\langle t^j, K^{j'}\rangle\langle y^k, E^{k'}\rangle,$$

where the pairing between $(\mathbb{C}[x]/(x^n))^{\underline{\mathrm{op}}}$ and $(\mathbb{C}[F]/(F^n))^{\underline{\mathrm{cop}}}$ implied by $\langle x, F\rangle = 1$ is $\langle x^{i(\underline{\mathrm{op}})}, F^{i'}\rangle = \delta_{i,i'}[i]_{q^{-1}}!$ while $\langle t^j, K^{j'}\rangle = q^{jj'}$ is implied by $\langle t, K\rangle = q$. The latter is the duality pairing in the Pontryagin sense in which $\mathbb{Z}_n$ is self-dual, and can be written as a usual dual pairing with the $\delta_j$. Equally well, $\left\{\dfrac{F^i\delta_j(K)E^k}{[i]_{q^{-1}}![k]_q!}\right\}_{0\le i,j,k<n}$ is a dual basis to $\{x^it^jy^k\}_{0\le i,j,k<n}$. $\square$

This applies even when $q = -1$, in that case as a self-duality pairing.

**Example 3.3.4.** If $q = -1$ then the double bosonisation $\mathfrak{u}_{-1}(sl_2) = B^{*\underline{\mathrm{cop}}}{\gtrdot\!\!\!\triangleleft} H {\triangleright\!\!\!<} B$ from

Lemma 3.3.2 has relations and coalgebra structure given by

$$E^2 = F^2 = 0, \quad K^2 = 1, \quad EF = FE, \quad KE = -EK, \quad KF = -FK,$$
$$\Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E$$

and is self-dual and strictly quasitriangular with

$$\mathcal{R} = (1 \otimes 1 - F \otimes E)\mathcal{R}_K, \quad \mathcal{R}_K = \frac{1}{2}(1 \otimes 1 + 1 \otimes K + K \otimes 1 - K \otimes K).$$

It is easy to check that this is not triangular, i.e $Q := \mathcal{R}_{21}\mathcal{R} = 1 \otimes 1 - E \otimes F - KF \otimes EK - EKF \otimes FKE \neq 1 \otimes 1$, and also not factorisable in the sense that the map $\mathfrak{u}_{-1}(sl_2)^* \to \mathfrak{u}_{-1}(sl_2)$ which sends $\phi \mapsto (\phi \otimes \mathrm{id})Q$ is not surjective (the element $FK \in \mathfrak{u}_{-1}(sl_2)$ is not in the image). On the other hand, Theorem 3.3.1 (1) gives us an isomorphic Hopf algebra by $x \mapsto F, y \mapsto E$ and $t \mapsto K$, so our Hopf algebra is self-dual, i.e., $\mathfrak{u}_{-1}(sl_2) \cong \mathfrak{c}_{-1}[SL_2]$. Note that $\mathfrak{u}_{-1}(sl_2)$ has the same dimension and the coalgebra structure of $u_{-1}(sl_2)$ but cannot be isomorphic, being noncommutative. One can also check that $\mathfrak{c}_{-1}[SL_2]$ is not isomorphic as a Hopf algebra to $c_{-1}[SL_2]$ and the latter, being noncocommutative, cannot be dual to $u_{-1}(sl_2)$.

## 3.4 Application to Hopf Algebra Fourier Transform

As a corollary of the above results, we briefly consider Hopf algebra Fourier transform between our double and codouble bosonisations. Recall from standard Hopf algebra theory, e.g. [19], that for a finite-dimensional Hopf algebra $H$ there is, up to scale, a unique right integral structure $\int : H \to k$ satisfying

$$\left(\int \otimes \mathrm{id}\right) \Delta h = \left(\int h\right) 1$$

for all $h \in H$. Such a right integral is the main ingredient for Fourier transform $\mathcal{F} : H \to H^*$. The following preliminary lemma is essentially well-known (see [19, Proposition, 1.7.7]), but for completeness we give the easier part that we need.

**Lemma 3.4.1.** *Let $\int, \int^*$ be right integrals on finite-dimensional Hopf algebras $H, H^*$ respectively and $\mu = \int(\int^*)$. The Fourier transform $\mathcal{F} : H \to H^*$ and adjunct $\mathcal{F}^*$ obey*

$$\mathcal{F}(h) := \sum_a \left(\int e_a h\right) f^a, \quad \mathcal{F}^*(\phi) := \sum_a e_a \Big(\int^* \phi f^a\Big), \quad \mathcal{F}^* \circ \mathcal{F} = \mu S,$$

*where $\{e_a\}$ is basis of $H$, $\{f^a\}$ is the dual basis of $H^*$. Hence $\mathcal{F}$ is invertible if $\mu \neq 0$.*

*Proof.* We write $\int^* = \Lambda^*$ when regarded as element in $H$. Then

$$\begin{aligned}\mathcal{F}^* \circ \mathcal{F}(h) =& e_a \Big(\int^* (\int e_b h) f^b f^a\Big) = (\int e_{a_{(1)}} h) e_{a_{(2)}} (\int^* f^a) \\ =& (\int \Lambda^*{}_{(1)} h_{(1)}) \Lambda^*{}_{(2)} h_{(2)} S h_{(3)} = (\int \Lambda^* h_{(1)}) S h_{(2)} = (\int \Lambda^*) S h = \mu S h.\end{aligned}$$

If $\mu \neq 0$ then this implies that $\mathcal{F}$ is injective and hence in our case invertible (with a bit more work [19] one can show that the inverse is $\mu^{-1} S^{-1} \mathcal{F}^*$). □

**Proposition 3.4.2.** *Let $q$ be a primitive $n$-th root of unity. The Fourier transform $\mathcal{F} : \mathfrak{c}_q[SL_2] \to \mathfrak{u}_q(sl_2)$ is invertible and given by*

$$\mathcal{F}(x^\alpha t^\beta y^\gamma) = \sum_{l=0}^{n-1} \frac{q^{-(l+\alpha)(1-\beta)+\beta(n-1-\gamma)}}{n[n-1-\alpha]_{q^{-1}}! [n-1-\gamma]_q!} F^{n-1-\alpha} K^l E^{n-1-\gamma}.$$

*Proof.* The right integral for $\mathfrak{c}_q[SL_2]$ is given by

$$\int x^\alpha t^\beta y^\gamma = \begin{cases} 1, \text{ if } \alpha = \gamma = n-1, \beta = 1 \\ 0, \text{ otherwise.} \end{cases}$$

This integral is equivalent in usual generators to $\int b^{n-1} c^{n-1} = 1$ and zero otherwise. Corollary 3.3.3 gives us the basis $\{e_a\} = \{x^i t^j y^k\}_{0 \leq i,j,k \leq n-1}$ of $\mathfrak{c}_q[SL_2]$ and the dual basis $\{f^a\} = \{\frac{F^i\delta_j(K)E^k}{[i]_{q^{-1}}!q^{j^2}[k]_q!}\}_{0\le i,j,k\le n-1}$ of $\mathfrak{u}_q(sl_2)$. Then

$$\begin{aligned}\mathcal{F}(x^\alpha t^\beta y^\gamma) &= \sum_{i,j,k=0}^{n-1}(\int x^i t^j y^k x^\alpha t^\beta y^\gamma)\frac{F^i\delta_j(K)E^k}{[i]_{q^{-1}}![k]_q!}\\ &= \sum_{i,j,k=0}^{n-1} q^{-\alpha j+\beta k}(\int x^{i+\alpha}t^{j+\beta}y^{k+\gamma})\frac{F^i\delta_j(K)E^k}{[i]_{q^{-1}}![k]_q!}\\ &=q^{-\alpha(1-\beta)+\beta(n-1-\gamma)}\frac{F^{2-\alpha}\delta_{1-\beta}(K)E^{n-1-\gamma}}{[n-1-\alpha]_{q^{-1}}![n-1-\gamma]_q!}\\ &=\sum_{l=0}^{n-1}\frac{q^{-(l+\alpha)(1-\beta)+\beta(n-1-\gamma)}}{(n-1+1)[n-1-\alpha]_{q^{-1}}![n-1-\gamma]_q!}F^{n-1-\alpha}K^l E^{n-1-\gamma}.\end{aligned}$$

The similar right integral of $\mathfrak{u}_q(sl_2)$ and resulting $\mu$ are

$$\int^* F^\alpha K^\beta E^\gamma = \begin{cases}1 & \text{if } \alpha=\gamma=n-1, \beta=1\\ 0 & \text{otherwise,}\end{cases} \qquad \mu = \frac{q^{-1}}{n[n-1]_{q^{-1}}![n-1]_q!},$$

which is nonzero. □

It appears to be a hard computational problem to give the general formula of the inverse Fourier transform, but one can compute it for specific cases.

**Example 3.4.3.** Let $q$ be a primitive cube root of unity. First, observe that for $\alpha, \beta = 0, 1, 2$, we have

$$\begin{aligned}[E^\alpha, F^\beta] = F^{\beta-1}([\alpha]_q[\beta]_q K - [\alpha]_{q^{-1}}[\beta]_{q^{-1}}K^{-1})E^{\alpha-1}\\ + F^{\beta-2}([2]_q K - [2]_{q^{-1}}K^{-1})(K-K^{-1})E^{\alpha-2}\end{aligned}$$

in $\mathfrak{u}_q(sl_2)$. Using this relation, we obtain

$$\begin{aligned}\mathcal{F}^*(F^\alpha K^\beta E^\gamma) =&\sum_{l=0}^{2}\frac{q^{\beta(2-\alpha)+(\gamma-l)(1-\beta)}}{3[2-\alpha]_{q^{-1}}![2-\gamma]_q!}x^{2-\alpha}t^l y^{2-\gamma}\\ &+\sum_{l=0}^{2}\frac{q^{\beta(l-\alpha-\gamma)}([\gamma]_q[3-\alpha]_q - q^{2(\gamma-l)+1}[\gamma]_{q^{-1}}[3-\alpha]_{q^{-1}})}{3[3-\alpha]_{q^{-1}}![3-\gamma]_q!}F^{3-\alpha}t^l y^{3-\gamma}\end{aligned}$$

$$- \frac{q^{2\beta+1}}{3[4-\alpha]_{q^{-1}}![4-\gamma]_q!} x^{4-\alpha} t^2 y^{4-\gamma}.$$

One can check that $\mathcal{F}^*\mathcal{F}(x^\alpha t^\beta y^\gamma) = \mu S(x^\alpha t^\beta y^\gamma)$, where $\mu = \frac{q^{-1}}{3[2]_{q^{-1}}![2]_q!} = \frac{q^2}{3}$ and

$$\begin{aligned} S(x^\alpha t^\beta y^\gamma) =& \frac{q^{\alpha\beta-\beta\gamma}}{[2-\alpha]_{q^{-1}}![\alpha]_{q^{-1}}![2-\gamma]_q![\gamma]_q!} x^\alpha t^{-\delta} y^\gamma \\ &+ \frac{q^{\alpha\beta-\beta\gamma}(q^{2\beta-2}-q^{\delta-2})}{[2-\alpha]_{q^{-1}}![\alpha]_{q^{-1}}![1-\gamma]_q![1+\gamma]_q!} x^{\alpha+1} t^{-\delta-1} y^{\gamma+1} \\ &- \frac{1+q^{\beta+1}+q^{2\beta+2}}{[2-\alpha]_{q^{-1}}![2+\alpha]_{q^{-1}}![2-\gamma]_q![2+\gamma]_q!} x^{2+\alpha} t^2 y^{2+\gamma}, \end{aligned}$$

where $\delta = \alpha + \beta + \gamma$.

**Example 3.4.4.** At $q = -1$, the Fourier transform in Proposition 3.4.2 combined with the self-duality in Example 3.3.4 becomes a Fourier transform operator $\mathfrak{c}_{-1}[SL_2] \to \mathfrak{c}_{-1}[SL_2]$. This has eigenvalues $\pm\frac{\imath}{\sqrt{2}}$ with multiplicity 2, $\pm\frac{(-1)^{1/4}}{\sqrt{2}}$ and $\pm\frac{(-1)^{3/4}}{\sqrt{2}}$ with multiplicity 1, and characteristic polynomial $f(x) = \frac{1}{16} + \frac{x^2}{4} + \frac{x^4}{2} + x^6 + x^8$. We also have

$$\mathcal{F}^*(F^a K^b E^c) = \frac{1}{2}\sum_{l=0}^{1}(-1)^{(1-b)(c-l)+b(1-a)} x^{1-a} t^l y^{1-c}$$

and one can check that $\mathcal{F}^{-1} = \mu^{-1}S^{-1}\mathcal{F}^*$ as in Lemma 3.4.1.

It is known that Fourier transform behaves well with respect to the coregular representation. This implies that it behaves well with respect to any covariant calculus. The following is known, see e.g. [35], but we include a short derivation in our conventions. In our case $H$ is finite-dimensional.

**Lemma 3.4.5.** *Let $\{e_a\}$ be a basis of $\Lambda^1$, $\{f^a\}$ a dual basis and define partial derivatives $\partial^a : H \to H$ by $\mathrm{d}h = \sum_a (\partial^a h) e_a$ and $\chi_a \in H^*$ by $\chi_a(h) = \langle f^a, \varpi\pi_\epsilon S^{-1}h\rangle$ for all $h \in H$. Then $\mathcal{F}(\partial^a h) = (\mathcal{F}h)\chi_a$ for all $h \in H$.*

*Proof.* Using the right-integral property, we have

$$\mathcal{F}(\partial^a h) = \mathcal{F}(h_{(1)})\langle f^a, \varpi\pi_\epsilon h_{(2)}\rangle = \sum_b \Big(\int e_{b(1)}h_{(1)}\Big) f^b \langle f^a, \varpi\pi_\epsilon((S^{-1}e_{b(3)})e_{b(2)}h_{(2)}\rangle$$

$$=\sum_b(\int e_{b(1)}h)f^b\langle f^a,\varpi\pi_\epsilon(S^{-1}e_{b(2)}\rangle=\sum_{b,c}(\int e_bh)f^bf^c\langle f^a,\varpi\pi_\epsilon(S^{-1}e_c\rangle=(\mathcal{F}h)\chi_a.$$

$\square$

**Example 3.4.6.** The 3D calculus c.f. [47] has left-invariant basic 1-forms $e_\pm, e_0$ with $e_\pm h = p^{|h|}he_\pm$ and $e_0h = p^{2|h|}he_0$ where $p = q^{-m}$ and $|\ |$ denotes a grading with $a, c$ grade 1 and $b, d$ grade -1 as a $\mathbb{Z}_n$-grading of $c_p[SL_2]$. Correspondingly for $\mathfrak{c}_q[SL_2]$, we have a calculus with $|x| = 0, |t| = |y| = 2$ and one can compute

$$\mathrm{d}x = q^{-m}te_-,\quad \mathrm{d}t = (1+q)\big(q(q^{-m}-q^m)tye_- + te_0\big),$$

$$\mathrm{d}y = (q^{-1}-1)^{-1}e_+ + (1+q)ye_0 + q(q^{-m}-q^m)y^2e_-,$$

which implies on a general monomial basis element that

$$\begin{aligned}\mathrm{d}(x^it^jy^k) =&(q^{-1}-1)^{-1}[k]_qx^it^jy^{k-1}e_+ + (1+q)[j+k]_{q^2}x^it^jy^ke_0\\ &+\Big(q(q^{-m}-q^m)[2j+k]_qx^it^jy^{k+1} + [i]_{q^{-1}}q^{-m+j+k}x^{i-1}t^{j+1}y^k\Big)e_-.\end{aligned}$$

We determine $\chi_a \in u_p(sl_2)$ from $\langle x^it^jy^k, \chi_a\rangle = \epsilon(\partial^a(x^it^jy^k))$ with the result

$$\chi_+ = \sum_{j=0}^{n-1}\frac{\delta_j(K)E}{q^{-1}-1} = \sum_{i,j=0}^{n-1}\frac{q^{-ij}}{n(q^{-1}-1)}K^iE = \frac{E}{q^{-1}-1},$$

$$\chi_0 = \sum_{j=0}^{n-1}(1+q)[j]_{q^2}\delta_j(K) = \sum_{i,j=0}^{n-1}\frac{q^{-ij}(1-q^{2j})}{n(1-q)}K^i = \frac{1-K^2}{1-q},$$

$$\chi_- = \sum_{j=0}^{n-1}q^{-m+j}F\delta_j(K) = \sum_{i,j=0}^{n-1}\frac{q^{-m}q^{j-ij}}{n}FK^i = q^{-m}FK.$$

These are versions of similar elements found for $\mathbb{C}_q[SU_2]$ with real $q$ in [47].

# Chapter 4

# Versions of $u_q(sl_3)$ and $c_q[SL_3]$ by (co)double bosonisation

As mentioned in the introduction, double bosonisation can in principle be used iteratively to construct all the $u_q(\mathfrak{g})$ [20, 28] and hence by making the corresponding codouble bosonisation at each step we will construct an appropriate $c_q[G]$.

In Section 4.1 we demonstrate this with

$$\mathfrak{u}_q(sl_3) = \mathfrak{c}_q^2 {>\!\!\!\triangleleft\kern-.33em\cdot\,} \widetilde{\mathfrak{u}_q(sl_2)} {\cdot\kern-.33em\triangleright\!\!\!<} \mathfrak{c}_q^2 = \mathfrak{c}_q^2 {>\!\!\!\triangleleft\kern-.33em\cdot\,} (\mathfrak{c}_q^1 {>\!\!\!\triangleleft\kern-.33em\cdot\,} \mathbb{C}\mathbb{Z}_n^2 {\cdot\kern-.33em\triangleright\!\!\!<} \mathfrak{c}_q^1) {\cdot\kern-.33em\triangleright\!\!\!<} \mathfrak{c}_q^2$$

where $\widetilde{\mathfrak{u}_q(sl_2)}$ is a central extension requiring an integer $\beta$ such that $\beta^2 = 3 \bmod n$, where we assume that $n = 2m + 1$ is odd, and $\mathfrak{c}_q^2$ denotes the reduced quantum-braided plane at the root of unity by making the generators nilpotent of order $n$. By setting $p = q^{-m}$, we will see that

$$\mathfrak{u}_q(sl_3) \cong \begin{cases} u_{q^{-m}}(sl_3) & m > 1, \\ (u_{q^{-m}}(sl_3)/\langle K_1 - K_2\rangle) \otimes (\mathbb{C}[g]/(g^n - 1)) & m = 1, \end{cases}$$

where the $m = 1$ case equates the two Cartan generators of the usual quantum group.

This quotient is necessarily quasitriangular by our construction whereas we are not clear if this is the case for $u_p(sl_3)$ itself when $q^3 = 1$.

We then construct the dual by $A = \widetilde{\mathfrak{c}_q[SL_2]}$ in Section 4.2 and similar quantum-braided planes now as Hopf algebras in its category comodules lead to a dual coquasitriangular Hopf algebra $\mathfrak{c}_q[SL_3]$. For $m > 1$ we show that this is isomorphic to the usual $c_{q^{-m}}[SL_3]$. Even at the second stage of $A = \widetilde{c_q(sl_2)}$, there are other potential choices for braided planes. We illustrate a non-classical choice where $A = \mathbb{C}_q[GL_2]$ is not finite-dimensional, $q$ is generic and $B = \mathbb{C}_q^{0|2}$ is the 'fermionic quantum-braided plane' in the category of $A$-comodules. This leads to an exotic but still coquasitriangular version of $\mathbb{C}_q[SL_3]$ with some matrix entries 'fermionic'.

We do not illustrate the inductive construction for cases beyond $\mathfrak{c}_q[SL_3]$ since the direct calculation becomes too complicated, but the method is not in doubt since one can always work backwards by deleting all root generators of a quantum group associated to a node in the Dynkin diagram (where the root vector contains the corresponding simple root) to give the sub-quantum group and finding the required quantum-braided plane for the induction step from the associated split projection of the $q$-Borel sub-Hopf algebra, c.f. [30] in the Lie bialgebra case. We then dualise the data for the corresponding codouble construction.

## 4.1 Construction of $\mathfrak{u}_q(sl_3)$ from $\mathfrak{u}_q(sl_2)$ by double bosonisation

The quantum group $u_q(sl_3)$ in more or less standard conventions is generated by $E_i, F_i, K_i$ for $i = 1, 2$, with, c.f. [14],

$$E_i^n = F_i^n = 0, \quad K_i^n = 1,$$

$$K_iK_j = K_jK_i, \quad E_iK_j = q^{a_{ij}}K_iE_j, \quad K_iF_j = q^{a_{ij}}F_jK_i, \quad [E_i, F_j] = \delta_{ij}(K_i - K_i^{-1}),$$

$$\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

where $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$. As before, we absorbed a factor $q - q^{-1}$ in the cross relation as a normalisation of $E_i$. We also require the $q$-Serre relations

$$E_i^2 E_j - (q + q^{-1})E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - (q + q^{-1})F_i F_j F_i + F_j F_i^2 = 0$$

for $i \neq j$. Note that $u_q(sl_2)$ appears as a sub-Hopf algebra generated by $E_1, F_1, K_1$.

Let $q$ be a primitive $n$-th root of unity with $n = 2m + 1$ and $p = q^{-m} = q^{\frac{1}{2}}$. Let $B = \mathfrak{c}_q^2$ be the algebra generated by $e_1, e_2$ with relation $e_2 e_1 = q^{-m} e_1 e_2$ in the category of right $\mathfrak{u}_q(sl_2)$-modules. The canonical left-action of $\mathfrak{u}_q(sl_2)$ on $B$ is given by

$$\begin{aligned}
\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft K &= \langle K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} q^{-m} & 0 \\ 0 & q^m \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\
\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft E &= \langle E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\
\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft F &= \langle F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},
\end{aligned} \tag{4.1.1}$$

where $\lambda = q^m - q^{-m}$. The duality between $\mathfrak{u}_q(sl_2)$ and $c_{q^{-m}}[SL_2]$ is the standard one when the former is identified with $u_{q^{-m}}(sl_2)$, or can be obtained from Corollary 3.3.3.

**Lemma 4.1.1.** *Let $q$ be a primitive $n$-th root of 1 with $n = 2m+1$ such that $\beta^2 = 3$ has a solution mod $n$. Let $H = \widetilde{\mathfrak{u}_q(sl_2)} = \mathfrak{u}_q(sl_2) \otimes \mathbb{C}_q[g]/(g^n - 1)$, and let $g$ act on $e_i$ by*

$$e_i \triangleleft g = q^{m\beta} e_i.$$

*Then $\mathfrak{c}_q^2$ is a braided Hopf algebra in the braided category of right $H$-modules with*

$$e_1^n = e_2^n = 0, \quad e_2 e_1 = q^{-m} e_1 e_2, \quad \underline{\Delta}(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \underline{\epsilon}(e_i) = 0, \quad \underline{S}(e_i) = -e_i,$$

$$\Psi(e_i \otimes e_i) = q e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = q^{-m} e_2 \otimes e_1, \quad \Psi(e_2 \otimes e_1) = q^{-m} e_1 \otimes e_2 + (q-1) e_2 \otimes e_1.$$

*Proof.* The quasitriangular structure of $\widetilde{\mathfrak{u}_q(sl_2)}$ is given by $\mathcal{R}_{\mathfrak{u}_q(sl_2)}\mathcal{R}_g$, where $\mathcal{R}_g = \frac{1}{n}\sum_{s,t=0}^{n-1} q^{-st} g^s \otimes g^t$ and $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ is given in Lemma 3.3.2. Thus, we can compute that

$$\Psi(e_i \otimes e_j) = q^{m^2\beta^2}(e_i \otimes e_j) \triangleleft \mathcal{R}_{\mathfrak{u}_q(sl_2)}.$$

This braiding is equal to the correctly normalised braiding in the statement (as needed for $\underline{\Delta}$ to extend as a homomorphism to the braided tensor product algebra) iff $m^2\beta^2 = m(m-1)$ mod $n$, or $m\beta^2 = m-1$ since any $m > 0$ is invertible mod $n$ (this is true for $m = 1$ and if $m > 1$ then $m$ and $2m+1$ are coprime). Thus the condition for $\mathfrak{c}_q^2$ to form a braided Hopf algebra in the category of $\widetilde{\mathfrak{u}_q(sl_2)}$-modules by an action of the stated form is $m(\beta^2 - 1) = -1 = 2m \mod n$, or $\beta^2 = 3 \mod n$. Some version of this lemma was largely in [6], working directly with $p = q^{-m}$. □

Here $\beta = 0$ is only possible for $m = 1$, i.e., $n = 3$. In this case $\mathfrak{c}_q^2$ is already a braided Hopf algebra in the category of $\mathfrak{u}_q(sl_2)$-modules without a central extension being needed. Otherwise, the least $n$ satisfying the condition is $n = 11$ with $\beta = 5$. For $n$ prime, $\beta$ exists if and only if $n = \pm 1 \mod 12$, see [46].

The dual $B^* = (\mathfrak{c}_q^2)^* \in {}_H M$ is generated by $f_1, f_2$ satisfying the same relations $f_2 f_1 = q^{-m} f_1 f_2$ and additive braided coproduct as $B$ but with the left action

$$\begin{aligned} &K \triangleright \begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} q^{-m} & 0 \\ 0 & q^m \end{pmatrix}, \quad g \triangleright f_i = q^{m\beta} f_i, \\ &E \triangleright \begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad F \triangleright \begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.1.2}$$

**Lemma 4.1.2.** *The quantum-braided planes $\mathfrak{c}_q^2$ and $(\mathfrak{c}_q^2)^*$ in Lemma 4.1.1 are dually paired by $\langle e_1^r e_2^s, f_1^{r'} f_2^{s'} \rangle = \delta_{r,r'}\delta_{s,s'}[r]_q![s]_q!$.*

*Proof.* It is not hard to see that $\langle e_i^r, f_i^{r'}\rangle = \delta_{r,r'}[r]_q!$ and this implies that

$$\begin{aligned}\langle e_1^r e_2^s, f_1^{r'} f_2^{s'}\rangle =&\langle e_1^r \otimes e_2^s, \sum_{r_1=0}^{r'}\sum_{s_1=0}^{s'} \begin{bmatrix} r' \\ r_1 \end{bmatrix}_q \begin{bmatrix} s' \\ s_1 \end{bmatrix}_q q^{-ms_1(r'-r_1)} f_1^{r_1} f_2^{s_1} \otimes f_1^{r'-r_1} f_2^{s'-s_1}\rangle \\ &= \langle e_1^r \otimes e_2^s, f_1^{r'} \otimes f_2^{s'}\rangle = \delta_{r,r'}\delta_{s,s'}[r]_q![s]_q!.\end{aligned}$$

□

In the double bosonisation, we read the generators $e_1, e_2$ of the quantum-braided plane $B = \mathfrak{c}_q^2$ as $E_{12}$ and $E_2$ respectively. Similarly, the generators $f_1, f_2$ of its dual quantum-braided plane $(\mathfrak{c}_q^2)^*$ are read as $F_{12}, F_2$ respectively. Also, we read the generators $E, F, K$ of $\mathfrak{u}_q(sl_2)$ as $E_1, F_1$ and $K_1$ so that

$$E_1^n = F_1^n = 0, \quad K_1^n = 1, \quad K_1E_1K_1^{-1} = q^{-1}E_1, \quad K_1F_1K_1^{-1} = qF_1, \quad [E_1, F_1] = K_1 - K_1^{-1}.$$

**Lemma 4.1.3.** *Suppose the setting of Lemma 4.1.1 with $n$ odd and $\beta^2 = 3$ solved mod $n$.*

1. *The double bosonisation of $\mathfrak{c}_q^2$, which we denote $\mathfrak{u}_q(sl_3)$, is generated by $E_i, F_i, K_1, g$ for $i = 1, 2$, with $E_1, F_1, K_1$ generating $\mathfrak{u}_q(sl_2)$ as a sub-Hopf algebra, and*

$$E_2K_1 = q^m K_1E_2, \quad E_2 g = q^{m\beta} gE_2, \quad K_1F_2 = q^m F_2K_1, \quad gF_2 = q^{m\beta}F_2 g,$$

$$[E_1, F_2] = [E_2, F_1] = 0, \quad [E_2, F_2] = K_1^m g^{m\beta} - K_1^{-m} g^{-m\beta},$$

$$\{E_i^2, E_j\} = (q^m + q^{-m})E_iE_jE_i, \quad \{F_i^2, F_j\} = (q^m + q^{-m})F_iF_jF_i; \quad i \neq j,$$

$$\Delta E_2 = 1 \otimes E_2 + E_2 \otimes K_1^m g^{m\beta}, \quad \Delta F_2 = F_2 \otimes 1 + g^{-m\beta}K_1^{-m} \otimes F_2,$$

$$\mathcal{R}_{\mathfrak{u}_q(sl_3)} = \frac{1}{n^2}\sum \frac{(-1)^{r+v+w} q^{vw-st-ab}}{[r]_{q^{-1}}![v]_{q^{-1}}![w]_{q^{-1}}!} F_{12}^v F_2^w F_1^r K_1^s g^a \otimes E_{12}^v E_2^w E_1^r K_1^t g^b,$$

*where we sum over $r, s, a, t, b, v, w$ from $0$ to $n-1$ and*

$$E_2E_1 = q^m E_1E_2 + \lambda E_{12}, \quad F_1F_2 = q^{-m}F_2F_1 + F_{12}; \quad \lambda = q^m - q^{-m}.$$

*2. If $n > 3$ and is not divisible by $3$ then $\mathfrak{u}_q(sl_3)$ is isomorphic to $u_{q^{-m}}(sl_3)$.*

*Proof.* (1) This is a direct computation using Theorem 3.1.1. First, we have that $E_2h = h_{(1)}(E_2{\triangleleft}h_{(2)})$ and $hF_2 = (h_{(1)}{\triangleright}F_2)h_{(2)}$ for all $h \in \widetilde{\mathfrak{u}_q(sl_2)}$ and using the actions mentioned above. Those not involving $E_{12}, F_{12}$ are as listed, while two more are regarded in the statement as definitions of $E_{12}, F_{12}$ in terms of the other generators. In this case the remaining cross relations

$$E_{12}K_1 = q^{-m}K_1E_{12}, \quad K_1F_{12} = q^{-m}F_{12}K_1, \quad E_{12}g = q^{m\beta}gE_{12}, \quad gF_{12} = q^{m\beta}F_{12}g$$

are all empty and can be dropped. Similarly, the first two of

$$[E_{12}, F_1] = K_1^{-1}E_2, \quad [E_1, F_{12}] = \lambda F_2K_1, \quad E_{12}E_1 = q^{-m}E_1E_{12}, \quad F_1F_{12} = q^mF_{12}F_1$$

are empty and can be dropped. The remaining two and the original quantum-braided plane relations $E_{12}E_2 = q^mE_2E_{12}, F_{12}F_2 = q^mF_2F_{12}$ are the four $q$-Serre relations stated for $i \neq i$. We next look at the cross relations between the two quantum-braided planes. For example,

$$[E_2, F_2] = \mathcal{R}^{(2)}\langle F_2, E_2 \triangleleft \mathcal{R}^{(1)}\rangle - \mathcal{R}^{-(1)}\langle \mathcal{R}^{-(2)} \triangleright F_2, E_2\rangle.$$

Putting in the form of $\mathcal{R}$ and $\mathcal{R}^{-1}$ gives the stated cross relation. One similarly has

$$[E_{12}, F_2] = -E_1K_1^mg^{m\beta}, \quad [E_2, F_{12}] = \lambda g^{-m\beta}K_1^{-m}F_1, \quad [E_{12}, F_{12}] = K_1^{-m}g^{m\beta} - K_1^mg^{-m\beta}$$

of which the first two are empty by a similar computation to the one above and the last is also empty by a more complicated calculation. In fact all these identities can be useful even though we do not include them in the defining relations. We also have

$$\begin{aligned}\Delta E_2 &= 1\otimes E_2 + \frac{1}{n^2}\sum \frac{(-1)^r}{[r]_{q^{-1}}!} q^{-st-ab}(E_2 \triangleleft F_1^r K_1^s g^a)\otimes E_1^r K_1^t g^b \\ &= 1\otimes E_2 + \frac{1}{n^2}\sum q^{-s(t-m)-a(b-m\beta)} E_2\otimes K_1^t g^b = 1\otimes E_2 + E_2\otimes K_1^m g^{m\beta},\end{aligned}$$

where we sum over $r,s,a,t,b$ from 0 to $n-1$. To compute $\Delta F_2$, we need

$$\mathcal{R}^{-1} = S\mathcal{R}^{(1)}\otimes \mathcal{R}^{(2)} = \frac{1}{n^2}\sum \frac{q^{-st-ab}}{[r]_{q^{-1}}!} g^{-a}K_1^{r-s}F_1^r\otimes E_1^r K_1^t g^b.$$

Only the first term contributes when acting on $F_2$,

$$\begin{aligned}\Delta F_2 &= F_2\otimes 1 + \frac{1}{n^2}\sum \frac{(-1)^r}{[r]_{q^{-1}}!} q^{-st-ab} g^{-a}K_1^{r-s}F_1^r\otimes E_1^r K_1^t g^b \triangleright F_2 \\ &= F_2\otimes 1 + \frac{1}{n^2}\sum q^{-t(s-m)-b(a-m\beta)} g^{-a}K_1^{-s}\otimes F_2 = F_2\otimes 1 + g^{-m\beta}K_1^{-m}\otimes F_2\end{aligned}$$

and similarly for $\Delta E_2$. One also has

$$\Delta E_{12} = 1\otimes E_{12} + E_{12}\otimes K_1^{-m}g^{m\beta} - E_2\otimes E_1 K_1^m g^{m\beta},$$

$$\Delta F_{12} = F_{12}\otimes 1 + g^{-m\beta}K_1^m\otimes F_{12} + \lambda g^{-m\beta}K_1^{-m}F\otimes F_2$$

which we did not state as $E_{12}, F_{12}$ are not generators. By Theorem 3.1.1 and Lemma 4.1.2, the quasitriangular structure of $\mathfrak{u}_q(sl_3)$ is

$$\mathcal{R}_{\mathfrak{u}_q(sl_3)} = \Big(\sum_{v,w=0}^{n-1} \frac{F_{12}^v F_2^w}{[v]_q![w]_q!}\otimes \underline{S}(E_{12}^v E_2^w)\Big)\mathcal{R}_{\mathfrak{u}_q(sl_2)}\mathcal{R}_g, \tag{4.1.3}$$

where $\mathcal{R}_{\mathfrak{u}_q(sl_2)}\mathcal{R}_g$ is explained in the proof of Lemma 4.1.1. By (2.3.1) for the braided-antipode, we find

$$\underline{S}(E_{12}^v E_2^w) = (-1)^{v+w} q^{\frac{v(v-1)+w(w-1)}{2}+vw} E_{12}^w E_2^v,$$

so that (4.1.3) becomes the expression stated.

(2) If $m>1$, we define $\varphi : u_{q^{-m}}(sl_3) \to \mathfrak{u}_q(sl_3)$ by

$$\varphi(E_i)=E_i, \quad \varphi(F_i)=F_i, \quad \varphi(K_1)=K_1, \quad \varphi(K_2)=K_1^m g^{m\beta}.$$

It is easy to see that $\varphi$ is an algebra and coalgebra map. In the other direction, when $m>1$, $\beta$ is invertible mod $n$ iff 3 is. We then define $\phi : \mathfrak{u}_q(sl_3) \to u_{q^{-m}}(sl_3)$ by

$$\phi(E_i)=E_i, \quad \phi(F_i)=F_i, \quad \phi(K_1)=K_1, \quad \varphi(g)=(K_1^{-m}K_2)^{\frac{1}{m\beta}},$$

which is clearly inverse to $\varphi$. □

We again write $p=q^{-m}$ so that $\mathfrak{u}_q(sl_3)$ is isomorphic to $u_p(sl_3)$ under our assumptions, where $n=33$ and $\beta=6$ is the first case excluded. The double bosonisation construction also gives $\{F_{12}^{i_1}F_2^{i_2}F_1^{i_3}K_1^{i_4}g^{i_5}E_1^{i_6}E_{12}^{i_7}E_2^{i_8}\}$ as a basis of $\mathfrak{u}_q(sl_3)$.

**Example 4.1.4.** As mentioned before, when $q$ is a primitive cubic root of unity i.e., when $\beta=0$, $\mathfrak{c}_q^2$ is already a braided Hopf algebra in the category of $\mathfrak{u}_q(sl_2)$-modules without an extension needed. Then Theorem 3.1.1 gives us a quasitriangular Hopf algebra, which we denote $\mathfrak{u}'_q(sl_3)$, generated by $E_i, F_i, K_1$ with $i=1,2$ with the relations and coproducts

$$E_iK_1=qK_1E_i, \quad K_1F_i=qF_iK_1, \quad [E_i,F_j]=\delta_{i,j}(K_1-K_1^{-1}),$$

$$\{E_i^2,E_j\}=(q+q^{-1})E_iE_jE_i, \quad \{F_i^2,F_j\}=(q+q^{-1})F_iF_jF_i; \quad i\neq j,$$

$$\Delta E_i=1\otimes E_i+E_i\otimes K_1, \quad \Delta F_i=F_i\otimes 1+K_1^{-1}\otimes F_i,$$

$$\mathcal{R}_H=\frac{1}{9}\sum\frac{(-1)^{r+v+w}q^{vw-st}}{[r]_{q^{-1}}![v]_{q^{-1}}![w]_{q^{-1}}!}F_{12}^vF_2^wF_1^rK_1^s\otimes E_{12}^vE_2^wE_1^rK_1^t,$$

where the sum is over $r,s,t,v,w$ from 0 to 2. This $\mathfrak{u}'_q(sl_3)$ is not isomorphic to $u_{q^{-1}}(sl_3)$ since we do not have the generator $K_2$. However, the element $K_1^{-1}K_2$ is central and group-like in $u_{q^{-1}}(sl_3)$ and $\mathfrak{u}'_q(sl_3)\cong u_{q^{-1}}(sl_3)/\langle K_1^{-1}K_2-1\rangle$. In addition, Lemma 4.1.3 still applies and $g$ is already group-like, and central when $\beta=0$. Therefore we have $\mathfrak{u}_q(sl_3)=\mathfrak{u}'_q(sl_3)\otimes\mathbb{C}_q[g]/(g^3-1)$ for $m=1$.

## 4.2 Construction of $\mathfrak{c}_q[SL_3]$ from $\mathfrak{c}_q[SL_2]$ by codouble bosonisation

Recall, see e.g. [19], that the coquasitriangular Hopf algebra $\mathbb{C}_q[SL_3]$ is generated by $\mathbf{t} = (t^i{}_j)$ for $i, j = 1, 2, 3$, with matrix-form of coproduct $\Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}$, and for $i < k$, $j < l$, the relations

$$[t^i{}_l, t^i{}_j]_q = 0, \quad [t^k{}_j, t^i{}_j]_q = 0, \quad [t^i{}_l, t^k{}_j] = 0, \quad [t^k{}_l, t^i{}_j] = \lambda t^i{}_l t^k{}_j,$$

$$\det{}_q(\mathbf{t}) := t^1{}_1(t^2{}_2t^3{}_3 - q^{-1}t^2{}_3t^3{}_2) - q^{-1}t^1{}_2(t^2{}_1t^3{}_3 - q^{-1}t^2{}_3t^3{}_1) + q^{-2}t^1{}_3(t^2{}_1t^3{}_2 - q^{-1}t^2{}_2t^3{}_1) = 1,$$

where $[a, b]_q := ba - qab$ and $\lambda = q - q^{-1}$. The reduced version is denoted by $c_q[SL_3]$ and has the additional relations

$$(t^i{}_j)^n = \delta_{ij}.$$

Throughout this section we limit ourselves to $q$ a primitive $n = 2m + 1$-th root of unity so that $\mathfrak{c}_q[SL_2] \cong c_{q^{-m}}[SL_2]$ according to Theorem 3.3.1. Since we only consider this case, it will be convenient to use the isomorphism to define new generators $a, b, c, d$ of $\mathfrak{c}_q[SL_2]$ related to our previous ones by $x = bd^{-1}, t = d^{-2}$ and $y = d^{-1}c/(q^m - q^{-m})$. Then we can benefit from both the matrix form of coproduct on the new set and the dual basis feature of the original set. We let $A = \widetilde{\mathfrak{c}_q[SL_2]} = \mathfrak{c}_q[SL_2] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^n - 1)$ be the central extension dual to $\widetilde{\mathfrak{u}_q(sl_2)} = \mathfrak{u}_q(sl_2) \otimes \mathbb{C}_q[g]/(g^n - 1)$. Here $\langle \varsigma, g \rangle = q$ and $\mathcal{R}(\varsigma, \varsigma) = q$ is the coquasitriangular structure on the central extension factor. Let $B$ be a quantum-braided plane $\mathfrak{c}_q^2$ as in Lemma 4.1.1 but viewed in the category of left comodules over $A$ with left coaction

$$\Delta_L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varsigma^{m\beta}, \tag{4.2.1}$$

where we now denote the generators $x_1, x_2$. In this case we will have

$$\Psi(x_i \otimes x_j) = q^{m^2\beta^2} R^j{}_k{}^i{}_l x_k \otimes x_l,$$

where $R$ was given in (3.3.3). We again require that $\beta^2 = 3 \mod n$ so that $q^{3m^2}R$ has the correct normalisation factor $q^{3m^2+m(m+1)} = q^{-m}$ in front of the matrix in (3.3.3), as needed to obtain a braided Hopf algebra. One also has, c.f. [19],

$$\underline{\Delta}(x_1^r x_2^s) = \sum_{r_1=0}^{r}\sum_{s_1=0}^{s} \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{-ms_1(r-r_1)} x_1^{r_1} x_2^{s_1} \otimes x_1^{r-r_1} x_2^{s-s_1}.$$

The dual $B^*$ was likewise explained in the previous section and is now taken with generators $y_i$ and regarded in the category of right comodules over $A$ with

$$\Delta_R \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \otimes \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}. \tag{4.2.2}$$

**Theorem 4.2.1.** *Let $n = 2m+1$ such that $\beta^2 = 3$ is solved mod $n$. Let $A = \widetilde{\mathfrak{c}_q[SL_2]} = \mathfrak{c}_q[SL_2] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^n - 1)$ regarded with generators $\varsigma, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. Let $B, B^*$ be quantum-braided planes with generators $x_i, y_i$ for $i = 1, 2$ as above.*

1. *The codouble bosonisation, denoted $\mathfrak{c}_q[SL_3]$, has cross relations and coproducts*

$$x_i y_j = y_j x_i, \quad x_i \varsigma = q^{m\beta} \varsigma x_i, \quad y_i \varsigma = q^{m\beta} \varsigma y_i, \quad \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} x_1 = \begin{pmatrix} q^{-1} x_1 \tilde{a} & q^{-1} x_1 \tilde{b} \\ q^m x_1 \tilde{c} & q^m x_1 \tilde{d} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} x_2 = \begin{pmatrix} q^m x_2 \tilde{a} + (q^{-1}-1) x_1 \tilde{c} & q^m x_2 \tilde{b} + (q^{-1}-1) x_1 \tilde{d} \\ q^{-1} x_2 \tilde{c} & q^{-1} x_2 \tilde{d} \end{pmatrix},$$

$$y_1 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} q\tilde{a} y_1 & q^{-m} \tilde{b} y_1 \\ q\tilde{c} y_1 & q^{-m} \tilde{d} y_1 \end{pmatrix}, \quad y_2 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} q^{-m} \tilde{a} y_2 + (q-1) \tilde{b} y_1 & q\tilde{b} y_2 \\ q^{-m} \tilde{c} y_2 + (q-1) \tilde{d} y_1 & q\tilde{d} y_2 \end{pmatrix},$$

$$\Delta x_1 = x_1\otimes 1+\sum_{r,s=0}^{n-1}(q-1)^{r+s}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \left(\tilde{a}y_1^r y_2^s\otimes x_1^{r+1}x_2^s+q^{-mr}\tilde{b}y_1^r y_2^s\otimes x_1^r x_2^{s+1}\right),$$

$$\Delta x_2 = x_2\otimes 1+\sum_{r,s=0}^{n-1}(q-1)^{r+s}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \left(\tilde{c}y_1^r y_2^s\otimes x_1^{r+1}x_2^s+q^{-mr}\tilde{d}y_1^r y_2^s\otimes x_1^r x_2^{s+1}\right),$$

$$\begin{aligned}\Delta y_1 =& 1\otimes y_1+\sum_{r,s=0}^{n-1}(q-1)^{r+s-1}q^{-r+ms+1}\begin{bmatrix} r+s-1 \\ s\end{bmatrix}_q y_1^r y_2^s\otimes x_1^{r-1}x_2^s\tilde{a}\\ &+\sum_{r,s=0}^{n-1}(q-1)^{r+s-1}q^{-r-s+1}\begin{bmatrix} r+s-1 \\ s-1\end{bmatrix}_q y_1^r y_2^s\otimes x_1^r x_2^{s-1}\tilde{c},\end{aligned}$$

$$\begin{aligned}\Delta y_2 =& 1\otimes y_2+\sum_{r,s=0}^{n-1}(q-1)^{r+s-1}q^{-r+ms+1}\begin{bmatrix} r+s-1 \\ s\end{bmatrix}_q y_1^r y_2^s\otimes x_1^{r-1}x_2^s\tilde{b}\\ &+\sum_{r,s=0}^{n-1}(q-1)^{r+s-1}q^{-r-s+1}\begin{bmatrix} r+s-1 \\ s-1\end{bmatrix}_q y_1^r y_2^s\otimes x_1^r x_2^{s-1}\tilde{d},\end{aligned}$$

$$\Delta\varsigma = \sum_{r,s=0}^{n-1}q^{-m\beta(r+s)}(q-1)^{r+s}\begin{bmatrix} r+2m\beta-1 \\ r\end{bmatrix}_q \begin{bmatrix} r+s+2m\beta-1 \\ s\end{bmatrix}_q \varsigma y_1^r y_2^s\otimes x_1^r x_2^s\varsigma,$$

$$\begin{aligned}\Delta\tilde{a} =& \sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{-r-s}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \tilde{a}y_1^r y_2^s\otimes\left(q^{-ms}[r+1]_q x_1^r x_2^s\tilde{a}+[s]_q x_1^{r+1}x_2^{s-1}\tilde{c}\right)\\ &+\sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{m(r+s)}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \tilde{b}y_1^r y_2^s\otimes\left(q^{-m}[r]_q x_1^{r-1}x_2^{s+1}\tilde{a}+q^s x_1^r x_2^s\tilde{c}\right),\end{aligned}$$

$$\begin{aligned}\Delta\tilde{b} =& \sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{-r-s}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \tilde{a}y_1^r y_2^s\otimes\left(q^{-ms}[r+1]_q x_1^r x_2^s\tilde{b}+[s]_q x_1^{r+1}x_2^{s-1}\tilde{d}\right)\\ &+\sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{m(r+s)}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \tilde{b}y_1^r y_2^s\otimes\left(q^{-m}[r]_q x_1^{r-1}x_2^{s+1}\tilde{b}+q^s x_1^r x_2^s\tilde{d}\right),\end{aligned}$$

$$\Delta\tilde{c} = \sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{-r-s}\begin{bmatrix} r+s \\ s\end{bmatrix}_q \tilde{c}y_1^r y_2^s\otimes\left(q^{-ms}[r+1]_q x_1^r x_2^s\tilde{a}+[s]_q x_1^{r+1}x_2^{s-1}\tilde{c}\right)$$

$$+\sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{m(r+s)}\begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{d}y_1^r y_2^s \otimes \left(q^{-m}[r]_q x_1^{r-1}x_2^{s+1}\tilde{a}+q^s x_1^r x_2^s \tilde{c}\right),$$

$$\Delta\tilde{d}=\sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{-r-s}\begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{c}y_1^r y_2^s \otimes \left(q^{-ms}[r+1]_q x_1^r x_2^s \tilde{b}+[s]_q x_1^{r+1}x_2^{s-1}\tilde{d}\right)$$

$$+\sum_{r,s=0}^{n-1}(q-1)^{r+s}q^{m(r+s)}\begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{d}y_1^r y_2^s \otimes \left(q^{-m}[r]_q x_1^{r-1}x_2^{s+1}\tilde{b}+q^s x_1^r x_2^s \tilde{d}\right).$$

2. *If $n>3$ and is not divisible by 3 then $\mathfrak{c}_q[SL_3]$ is isomorphic to $c_{q^{-m}}[SL_3]$ with its standard coquasitriangular structure.*

*Proof.* (1) From $x_2 \cdot_{\underline{\mathrm{op}}} x_1 = qx_1 \cdot_{\underline{\mathrm{op}}} x_2$, we work inductively and find that

$$x_1^{r(\underline{\mathrm{op}})}\cdot_{\underline{\mathrm{op}}} x_2^{s(\underline{\mathrm{op}})} = q^{\frac{-(r+s)(r+s-1)}{2}}x_1^r x_2^s,\quad \overline{S}(x_1^{r(\underline{\mathrm{op}})}\cdot_{\underline{\mathrm{op}}} x_2^{s(\underline{\mathrm{op}})}) = (-1)^{r+s}q^{\frac{-(r+s)(r+s-1)}{2}}x_1 x_2$$

where $x_1^{r(\underline{\mathrm{op}})}$ means $x_1 \cdot_{\underline{\mathrm{op}}} x_1 \cdot_{\underline{\mathrm{op}}} \cdots$ $r$-times. We also need that

$$\Delta_L(x_1^r)=\sum_{r_1=0}^{r}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \tilde{a}^{r_1}\tilde{b}^{r-r_1}\otimes x_1^{r_1}x_2^{r-r_1},\quad \Delta_L(x_2^s)=\sum_{s_1=0}^{s}\begin{bmatrix} s \\ s_1 \end{bmatrix}_q \tilde{c}^{r_1}\tilde{d}^{s-s_1}\otimes x_1^{s_1}x_2^{s-s_1}$$

and that $\zeta$ commutes with $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$. Then computation from Theorem 3.2.2 gives

$$x_i y_j = y_j x_i,\quad x_i\varsigma = q^{m\beta}\varsigma x_i,\quad y_i\varsigma = q^{m\beta}\varsigma y_i,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_1 = \begin{pmatrix} q^{-m^2}x_1 a & q^{-m^2}x_1 b \\ q^{m^2}x_1 c & q^{m^2}x_1 d \end{pmatrix},\quad y_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{m^2}ay_1 & q^{-m^2}by_1 \\ q^{m^2}cy_1 & q^{-m^2}dy_1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_2 = \begin{pmatrix} q^{m^2}(x_2 a+(q^m-q^{-m})x_1 c) & q^{m^2}(x_2 b+(q^m-q^{-m})x_1 d) \\ q^{-m^2}x_2 c & q^{-m^2}x_2 d \end{pmatrix},$$

$$y_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-m^2}(ay_2-(q^m-q^{-m})by_1) & q^{m^2}by_2 \\ q^{-m^2}(cy_2-(q^m-q^{-m})dy_1) & q^{m^2}dy_2 \end{pmatrix}$$

and hence the relations stated. The algebra generated by $x_i, y_i, \varsigma, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ is $n^8$ dimensional as required for these to be all the relations. For the coproduct, we use Lemma 4.1.2 to provide a basis and dual basis of $\mathfrak{c}_q^2$ and $(\mathfrak{c}_q^2)^*$. Then

$$\begin{aligned}\Delta x_1 =& x_1\otimes 1\\ &+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+\frac{(r_1+s_1)(r_1+s_1+1)}{2}+r_1+s_1}\\ &\qquad\qquad \times\frac{\tilde{a}y_1^ry_2^s}{[r]_q![s]_q!}\otimes x_1^{r+1}x_2^s\\ &+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}+s_1-mr}\\ &\qquad\qquad \times(1+[r_1]_q(q-1))\frac{\tilde{b}y_1^ry_2^s}{[r]_q![s]_q!}\otimes x_1^rx_2^{s+1}\end{aligned}$$

and similarly for $\Delta x_2$. Likewise,

$$\begin{aligned}\Delta y_1 =& 1\otimes y_1\\ &+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+\frac{(r_1+s_1-1)(r_1+s_1-2)}{2}-r+ms+r_1+s_1}\\ &\qquad\qquad \times[r_1]_q\frac{y_1^ry_2^s}{[r]_q![s]_q!}\otimes x_1^{r-1}x_2^s\tilde{a}\\ &+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{r_1+s_1-r-s+\frac{(r_1+s_1)(r_1+s_1-1)}{2}}(-1)^{r+s-r_1-s_1}\\ &\qquad\qquad \times([s_1]_q+[r_1]_q[s-s_1]_q)\frac{y_1^ry_2^s}{[r]_q![s]_q!}\otimes x_1^rx_2^{s-1}\tilde{c}\end{aligned}$$

and similarly for $\Delta y_2$. Next, we have

$$\Delta\varsigma = \sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{s_1(r-r1)+m\beta(2r_1+2s_1-r-s)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}}(-1)^{r+s-r_1-s_1}$$
$$\times \frac{\varsigma y_1^r y_2^s}{[r]_q![s]_q!}\otimes x_1^r x_2^s\varsigma,$$

$$\Delta\tilde{a} = \sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+2r_1+s_1-r+ms+\frac{(r_1+s1)(r_1+s_1-1)}{2}}$$
$$\times \frac{\tilde{a}y_1^r y_2^s}{[r]_q![s]_q!}\otimes x_1^r x_2^s\tilde{a}$$
$$+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [s-s_1]_q(-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+2r_1+2s_1-r-s+\frac{(r_1+s1)(r_1+s_1-1)}{2}}$$
$$\times (1-q)\frac{\tilde{a}y_1^r y_2^s}{[r]_q![s]_q!}\otimes x_1^{r+1}x_2^{s-1}\tilde{c}$$
$$+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [r_1]_q(-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+r1+s1+mr+ms+\frac{(r_1+s1)(r_1+s_1-1)}{2}}$$
$$\times (q^{-m}-q^m)\frac{\tilde{b}y_1^r y_2^s}{[r]_q![s]_q!}\otimes x_1^{r-1}x_2^{s+1}\tilde{a}$$
$$+\sum_{r,s=0}^{n-1}\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}\begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1}q^{s_1(r-r_1)+\frac{(r_1+s1)(r_1+s_1-1)}{2}+r_1+2s_1+mr-s}$$
$$\times (1-[r_1]_q[s-s_1]_q(q-1)^2)\frac{\tilde{b}y_1^r y_2^s}{[r]_q![s]_q!}\otimes x_1^r x_2^s\tilde{c}$$

and similarly for $\Delta\tilde{b}, \Delta\tilde{c}, \Delta\tilde{d}$. The stated coproducts follow from the $q$-identity

$$\sum_{r_1=0}^{r}\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}q^{sr_1}}{[r_1]_q![r-r_1]_q!} = (q-1)^r\begin{bmatrix} r+s \\ r \end{bmatrix}_q \tag{4.2.3}$$

for all $r, s$ (of which (3.3.2) are special cases) and further calculation, which will be given in the Appendix.

(2) If $n > 3$ and is not divisible by 3 then $\beta$ is invertible mod $n$. We define $\varphi$ :

$c_{q^{-m}}[SL_3] \to \mathfrak{c}_q[SL_3]$ by

$$\varphi(t^1{}_1) = a\varsigma^{\frac{m}{\beta}} + \lambda x_1 \varsigma^{\frac{1}{\beta}} y_1, \quad \varphi(t^1{}_2) = b\varsigma^{\frac{m}{\beta}} + \lambda x_1 \varsigma^{\frac{1}{\beta}} y_2, \quad \varphi(t^1{}_3) = x_1 \varsigma^{\frac{1}{\beta}},$$

$$\varphi(t^2{}_1) = c\varsigma^{\frac{m}{\beta}} + \lambda x_2 \varsigma^{\frac{1}{\beta}} y_1, \quad \varphi(t^2{}_2) = d\varsigma^{\frac{m}{\beta}} + \lambda x_2 \varsigma^{\frac{1}{\beta}} y_2, \quad \varphi(t^2{}_3) = x_2 \varsigma^{\frac{1}{\beta}},$$

$$\varphi(t^3{}_1) = \lambda \varsigma^{\frac{1}{\beta}} y_1, \quad \varphi(t^3{}_2) = \lambda \varsigma^{\frac{1}{\beta}} y_2, \quad \varphi(t^3{}_3) = \varsigma^{\frac{1}{\beta}},$$

where $\lambda = q^m - q^{-m}$. A tedious calculation shows that this extends as an algebra map and is a coalgebra map. In the other direction, we define $\phi : \mathfrak{c}_q[SL_3] \to c_{q^{-m}}[SL_3]$ by

$$\phi(\varsigma) = (t^3{}_3)^{\beta}, \quad \phi(x_1) = t^1{}_3 (t^3{}_3)^{-1}, \quad \phi(x_2) = t^2{}_3 (t^3{}_3)^{-1},$$

$$\phi(y_1) = \lambda^{-1} (t^3{}_3)^{-1} t^3{}_1, \quad \phi(y_2) = \lambda^{-1} (t^3{}_3)^{-1} t^3{}_2,$$

$$\phi(a) = t^1{}_1 (t^3{}_3)^{-m} - q^m t^1{}_3 t^3{}_1 (t^3{}_3)^m, \quad \phi(b) = t^1{}_2 (t^3{}_3)^{-m} - q^m t^1{}_3 t^3{}_2 (t^3{}_3)^m,$$

$$\phi(c) = t^2{}_1 (t^3{}_3)^{-m} - q^m t^2{}_3 t^3{}_1 (t^3{}_3)^m, \quad \phi(d) = t^2{}_2 (t^3{}_3)^{-m} - q^m t^2{}_3 t^3{}_2 (t^3{}_3)^m$$

as inverse to $\varphi$. Although one can verify these matters directly, the map $\varphi$ was obtained as adjoint to the isomorphism $u_{q^{-m}}(sl_3) \to \mathfrak{u}_q(sl_3)$ in part (2) of Lemma 4.1.3 as follows. The standard duality between $u_{q^{-m}}(sl_3)$ and $c_{q^{-m}}[SL_3]$ is by

$$\langle \mathbf{t}, F_1 \rangle = \mathbf{e}_{12}, \quad \langle \mathbf{t}, F_2 \rangle = \mathbf{e}_{23}, \quad \langle \mathbf{t}, F_{12} \rangle = \mathbf{e}_{13}, \quad \langle \mathbf{t}, E_1 \rangle = \lambda \mathbf{e}_{21}, \quad \langle \mathbf{t}, E_2 \rangle = \lambda \mathbf{e}_{32},$$

$$\langle \mathbf{t}, E_{12} \rangle = \lambda \mathbf{e}_{31}, \quad \langle \mathbf{t}, K_1 \rangle = \begin{pmatrix} q^{-m} & 0 & 0 \\ 0 & q^m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \mathbf{t}, K_2 \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-m} & 0 \\ 0 & 0 & q^m \end{pmatrix},$$

where $\mathbf{e}_{ij}$ is an elementary matrix with entry 1 in $(i,j)$-position and 0 elsewhere. The duality between $\mathfrak{u}_q(sl_3)$ and $\mathfrak{c}_q[SL_3]$ is part of our construction with a natural basis of $\mathfrak{c}_q[SL_3]$ built from bases of $\mathfrak{c}_q^2, (\mathfrak{c}_q^2)^*$ and $\widetilde{\mathfrak{c}_q[SL_2]} = \mathfrak{c}_q[SL_2] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^n - 1)$. The first tensor factor here has a basis of monomials in $x, t, y$ by Theorem 3.3.1. Therefore we have $\{x_1^{i_1} x_2^{i_2} x^{i_3} t^{j_1} \varsigma^{j_2} y^{k_1} y_1^{k_2} y_2^{k_3}\}$ as a basis of $\mathfrak{c}_q[SL_3]$ essentially dual to the PBW basis of $\mathfrak{u}_q(sl_3)$ in the sense

$$\langle x_1^{i_1}x_2^{i_2}x^{i_3}t^{j_1}\varsigma^{j_2}y^{k_1}y_1^{k_2}y_2^{k_3}, F_{12}^{i'_1}F_2^{i'_2}F_1^{i'_3}K_1^{j'_1}g^{j'_2}E_1^{k'_1}E_{12}^{k'_2}E_2^{k'_3}\rangle$$
$$= \delta_{i_1i'_1}\delta_{i_2i'_2}\delta_{i_3i'_3}\delta_{k_1k'_1}\delta_{k_2k'_2}\delta_{k_3k'_3}[i_1]_{q^{-1}}![i_2]_{q^{-1}}![i_3]_{q^{-1}}!q^{j_1j'_1+j_2j'_2}[k_1]_q![k_2]_q![k_3]_q!.$$

This is the dual basis result for $\mathfrak{u}_q(sl_3)$ and $\mathfrak{c}_q[SL_3]$ analogous to Corollary 3.3.3 in the $sl_2$ case. Hence the coefficients of $\varphi(t^i{}_j)$ in this basis of $\mathfrak{c}_q[SL_3]$ will be picked out by evaluation against the dual basis $F_{12}^{i_1}F_2^{i_2}F_1^{i_3}\delta_{j_1}(K_1)\delta_{j_2}(g)E_1^{k_1}E_{12}^{k_2}E_2^{k_3}$, where $\delta_j(K_1), \delta_j(g)$ are defined as in Corollary 3.3.3. These values are given by the matrix representation as above except that we still need the matrix representation of $g$. From Lemma 4.1.3 we recall that that $\mathfrak{u}_q(sl_3) \cong u_{q^{-m}}(sl_3)$ with $g \mapsto (K^{-m}K_2)^{\frac{1}{m\beta}}$, hence we have $\langle \mathbf{t}, g\rangle = \mathrm{diag}(q^{\frac{m}{\beta}}, q^{\frac{m}{\beta}}, q^{\frac{1}{\beta}})$. Write $J = F_{12}^{i_1}F_2^{i_2}F_1^{i_3}\delta_{j_1}(K_1)\delta_{j_2}(g)E_1^{k_1}E_{12}^{k_2}E_2^{k_3}$, then this gives

$$\begin{aligned}
\langle\varphi(t^1{}_1), J\rangle =&\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,-m}\delta_{j_2,\frac{m}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}\\
&+\lambda\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,1}\delta_{j_1,m}\delta_{j_2,\frac{m}{\beta}}\delta_{k_1,1}\delta_{k_2,0}\delta_{k_3,0}\\
&+\lambda\delta_{i_1,1}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,1}\delta_{k_3,0}\\
\langle\varphi(t^1{}_2), J\rangle =&\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,1}\delta_{j_1,m}\delta_{j_2,\frac{m}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}\\
&+\lambda\delta_{i_1,1}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,1}\\
\langle\varphi(t^1{}_3), J\rangle =&\delta_{i_1,1}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}\\
\langle\varphi(t^2{}_1), J\rangle =&\lambda\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,m}\delta_{j_2,\frac{m}{\beta}}\delta_{k_1,1}\delta_{k_2,0}\delta_{k_3,0}\\
&+\lambda\delta_{i_1,0}\delta_{i_2,1}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,1}\delta_{k_3,0}\\
\langle\varphi(t^2{}_2), J\rangle =&\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,m}\delta_{j_2,\frac{m}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}\\
&+\lambda\delta_{i_1,0}\delta_{i_2,1}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,1}\\
\langle\varphi(t^2{}_3), J\rangle =&\delta_{i_1,0}\delta_{i_2,1}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}\\
\langle\varphi(t^3{}_1), J\rangle =&\lambda\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,1}\delta_{k_3,0}\\
\langle\varphi(t^3{}_2), J\rangle =&\lambda\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,1}\\
\langle\varphi(t^3{}_3), J\rangle =&\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}\delta_{j_1,0}\delta_{j_2,\frac{1}{\beta}}\delta_{k_1,0}\delta_{k_2,0}\delta_{k_3,0}.
\end{aligned}$$

We then convert to the $a, b, c, d$ generators as discussed.

Finally, the coquasitriangular structure of $\mathfrak{c}_q[SL_3]$ computed using Lemma 3.2.7 and pulled back to the $c_{q^{-m}}[SL_3]$ generators is $\mathcal{R}(\varphi(t^i{}_j), \varphi(t^k{}_l)) = R^I{}_J$, where $I = (i,k)$, $J = (j,l)$ are taken in lexicographic order $(1,1), (1,2), \cdots, (3,3)$ and

$$R^I{}_J = q^{\frac{m}{3}} \begin{pmatrix} q^{-m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q^{-m}-q^m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q^{-m}-q^m & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q^{-m}-q^m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-m} \end{pmatrix},$$

which is the standard coquasitriangular structure on the generators of $c_p[SL_3]$ given in [20] when specialised to the root of unity $p = q^{-m}$. □

**Remark 4.2.2.** In the case (2) of the theorem above, we can identify $\widetilde{\mathfrak{c}_q[SL_2]} \cong c_{q^{-m}}[GL_2]$ by sending the four matrix generators of the latter to $\tilde{a} = a\varsigma^{m\beta}, \tilde{b} = b\varsigma^{m\beta}, \tilde{c} = c\varsigma^{m\beta}, \tilde{d} = d\varsigma^{m\beta}$. The $q$-determinant $D$ maps to $\varsigma^{2m\beta}$. The converse direction is clear since $\beta$ is invertible mod $n$ when $n > 3$ and not divisible by 3, so we can write $\varsigma = D^{\frac{1}{2m\beta}}$.

**Example 4.2.3.** At $q^3 = 1$, $\mathfrak{c}_q^2$ is already a braided Hopf algebra in the category of $\mathfrak{c}_q[SL_2]$-comodules without a central extension. Therefore we can apply Theorem 3.2.2 and obtain a Hopf algebra, which we denote $\mathfrak{c}'_q[SL_3]$, generated by $x_i, y_i, a, b, c, d$ with the additional cross relations and coproducts

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_1 = \begin{pmatrix} q^{-1}x_1a & q^{-1}x_1b \\ qx_1c & qx_1d \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_2 = \begin{pmatrix} qx_2a + (q^{-1}-1)x_1c & qx_2b + (q^{-1}-1)x_1d \\ q^{-1}x_2c & q^{-1}x_2d \end{pmatrix},$$

$$y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qay_1 & q^{-1}by_1 \\ qcy_1 & q^{-1}dy_1 \end{pmatrix}, \quad y_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}ay_2 + (q-1)by_1 & qby_2 \\ q^{-1}cy_2 + (q-1)dy_1 & qdy_2 \end{pmatrix},$$

$$\begin{aligned}
\Delta x_1 =& x_1 \otimes 1 + a \otimes x_1 + b \otimes x_2 + \lambda ay_1 \otimes x_1^2 + \lambda by_2 \otimes x_2^2 + \lambda ay_2 \otimes x_1x_2 \\
&+ \lambda q^2 by_1 \otimes x_1x_2 + \lambda^2 ay_2^2 \otimes x_1x_2^2 + \lambda^2 qby_1^2 \otimes x_1^2x_2 \\
&+ \lambda^2[2]_q ay_1y_2 \otimes x_1^2x_2 + \lambda^2[2]_q q^2 by_1y_2 \otimes x_1x_2^2 \\
, \Delta y_1 =& 1 \otimes y_1 + y_1 \otimes a + y_2 \otimes c + \lambda q^2 y_1^2 \otimes x_1a + \lambda q^2 y_2^2 \otimes x_2c + \lambda qy_1y_2 \otimes x_2a \\
&+ \lambda q^2 y_1y_2 \otimes x_1c + \lambda^2 q^2 y_1y_2^2 \otimes x_2^2a + \lambda^2[2]_q qy_1y_2^2 \otimes x_1x_2c \\
&+ \lambda^2[2]_q y_1^2y_2 \otimes x_1x_2a + \lambda^2 qy_1^2y_2 \otimes x_1^2c \\
\Delta a =& a \otimes a + b \otimes c + \lambda[2]_q q^2 ay_1 \otimes x_1a + \lambda qay_2 \otimes x_2a + \lambda q^2 ay_2 \otimes x_1c \\
&+ \lambda[2]_q q^{-1} by_2 \otimes x_2c + \lambda by_1 \otimes x_2a + \lambda qby_1 \otimes x_1c + \lambda^2 q^2 ay_2^2 \otimes x_2^2a \\
&+ \lambda^2 q^2 by_1^2 \otimes x_1^2c + \lambda^2[2]_q qay_2^2 \otimes x_1x_2c + \lambda^2[2]_q qby_1^2 \otimes x_1x_2a \\
&+ \lambda^2[2]_q^2 ay_1y_2 \otimes x_1x_2a + \lambda^2[2]_q^2 by_1y_2 \otimes x_1x_2c \\
&+ \lambda^2[2]_q qay_1y_2 \otimes x_1^2c + \lambda^2[2]_q qby_1y_2 \otimes x_2^2a,
\end{aligned}$$

and similarly for the remaining coproducts. Here $\lambda = q-1$. This $\mathfrak{c}'_q[SL_3]$ is dual to $\mathfrak{u}'_q(sl_3)$ in Example 4.1.4 and it is not isomorphic to $c_{q^{-1}}[SL_3]$, but rather to a sub-Hopf algebra by $\phi : \mathfrak{c}'_q[SL_3] \hookrightarrow c_{q^{-1}}[SL_3]$ with

$$\phi(x_1) = t^1{}_3(t^3{}_3)^{-1}, \quad \phi(x_2) = t^2{}_3(t^3{}_3)^{-1}, \quad \phi(y_1) = -\frac{t^3{}_1(t^3{}_3)^{-1}}{\lambda}, \quad \phi(y_2) = -\frac{t^3{}_2(t^3{}_3)^{-1}}{\lambda},$$

$$\phi(a) = t^1{}_1(t^3{}_3)^{-1} - t^1{}_3(t^3{}_3)^{-1}t^3{}_1(t^3{}_3)^{-1}, \quad \phi(b) = t^1{}_2(t^3{}_3)^{-1} - t^1{}_3(t^3{}_3)^{-1}t^3{}_2(t^3{}_3)^{-1},$$

$$\phi(c) = t^2{}_1(t^3{}_3)^{-1} - t^2{}_3(t^3{}_3)^{-1}t^3{}_1(t^3{}_3)^{-1}, \quad \phi(d) = t^2{}_2(t^3{}_3)^{-1} - t^2{}_3(t^3{}_3)^{-1}t^3{}_2(t^3{}_3)^{-1}.$$

Moreover, $\mathfrak{c}'_q[SL_3]$ is a coquasitriangular Hopf algebra by Lemma 3.2.7. Writing $s^1{}_1 = a, s^1{}_2 = b, s^2{}_1 = c, s^2{}_2 = d$ for the matrix form of the generators of $\mathfrak{c}_q[SL_2]$, the coquasitriangular structure of $\mathfrak{c}'_q[SL_3]$ comes out as

$$\mathcal{R}(s^i{}_j, s^k{}_l) = R^i{}_j{}^k{}_l, \quad \mathcal{R}(x_i, y_j) = -\delta_{i,j}, \quad \mathcal{R}(x_i, x_j) = \mathcal{R}(y_i, y_j) = \mathcal{R}(y_i, x_j) = 0,$$

$$\mathcal{R}(x_i, s^j{}_k) = \mathcal{R}(y_i, s^j{}_k) = \mathcal{R}(s^i{}_j, x_k) = \mathcal{R}(s^i{}_j, y_k) = 0,$$

$$\mathcal{R}(x_i s^j{}_k, s^u{}_v y_w) = -\delta_{w_1,i} R^j{}_{j_1}{}^{w_1}{}_w R^{j_1}{}_k{}^u{}_v, \quad \mathcal{R}(s^i{}_j y_k, x_u s^v{}_w) = 0,$$

where $R$ is as in (3.3.3) with $m = 1$. Theorem 4.2.1 (1) still applies at $q^3 = 1$ with $\beta = 0$ giving that $\varsigma$ is central and group-like in $\mathfrak{c}_q[SL_3]$ and that $\mathfrak{c}_q[SL_3] \cong \mathfrak{c}'_q[SL_3] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^3 - 1)$.

## 4.3 Fermionic version of $\mathbb{C}_q[SL_3]$

Here we similarly apply codouble bosonisation but this time to obtain a part-fermionic version of $\mathbb{C}_q[SL_3]$ by using the fermionic quantum-braided plane. We no longer work at roots of unity but rather with $q$ generic and also, in the spirit of Remark 4.2.2, we take as our middle Hopf algebra $A = \mathbb{C}_q[GL_2]$, the coquasitriangular Hopf algebra generated by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ with the same $q$-commutation relations and coalgebra structure as $\mathbb{C}_q[SL_2]$, but with $D = \tilde{a}\tilde{d} - q^{-1}\tilde{b}\tilde{c} = \tilde{d}\tilde{a} - q\tilde{b}\tilde{c}$ inverted. The antipode and coquasitriangular structure are given in matrix form by

$$S\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = D^{-1}\begin{pmatrix} \tilde{d} & -q\tilde{b} \\ -q^{-1}\tilde{c} & \tilde{a} \end{pmatrix}, \quad R = -q^{-1}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

In fact the normalisation of $R$ here can be chosen freely (there is a 1-parameter family of such quasitriangular structures on this Hopf algbra) which we have fixed so that we have $B = \mathbb{C}_q^{0|2} \in {}^A\mathcal{M}$ as a fermionic quantum-braided plane generated by $e_1, e_2$ with the relations and coproduct and braiding

$$e_i^2=0,\quad e_2e_1+q^{-1}e_1e_2=0,\quad \underline{\Delta}e_i=e_i\otimes 1+1\otimes e_i,\quad \underline{\epsilon}e_i=0,\quad \underline{S}e_i=-e_i,$$

$$\Psi(e_i\otimes e_i)=-e_i\otimes e_i,\quad \Psi(e_1\otimes e_2)=-q^{-1}e_2\otimes e_1,\quad \Psi(e_2\otimes e_1)=-q^{-1}e_1\otimes e_2-(1-q^{-2})e_2\otimes e_1.$$

This has a left $\mathbb{C}_q[GL_2]$-coaction as in (4.2.1). Similarly, its dual $B^*=(\mathbb{C}_q^{0|2})^*$ lives in the category of right $\mathbb{C}_q[GL_2]$-comodules with coaction as in (4.2.2).

**Proposition 4.3.1.** *Let $q\in\mathbb{C}^*$ not be a root of unity. The codouble bosonisation $B^{\underline{\mathrm{op}}}{\blacktriangleright\!\!\!<}A{\blacktriangleright\!\!\!<}B^*$ with the above $B, A, B^*$ is a coquasitriangular Hopf algebra $\mathbb{C}_q^{fer}[SL_3]$ generated by $e_i, f_i$ for $i=1,2$ and $\tilde a,\tilde b,\tilde c,\tilde d, D, D^{-1}$, with cross relations and coproducts*

$$f_ie_j=e_jf_i,\qquad \begin{pmatrix}\tilde a&\tilde b\\ \tilde c&\tilde d\end{pmatrix}e_1=\begin{pmatrix}-e_1\tilde a&-e_1\tilde b\\ -qe_1\tilde c&-qe_1\tilde d\end{pmatrix},$$

$$\begin{pmatrix}\tilde a&\tilde b\\ \tilde c&\tilde d\end{pmatrix}e_2=\begin{pmatrix}-qe_2\tilde a-(1-q^2)e_1\tilde c&-qe_2\tilde b-(1-q^2)e_1\tilde d\\ -e_2\tilde c&-e_2\tilde d\end{pmatrix},$$

$$f_1\begin{pmatrix}\tilde a&\tilde b\\ \tilde c&\tilde d\end{pmatrix}=\begin{pmatrix}-\tilde af_1&-q^{-1}\tilde bf_1\\ -\tilde cf_1&-q^{-1}\tilde df_1\end{pmatrix},\quad f_2\begin{pmatrix}\tilde a&\tilde b\\ \tilde c&\tilde d\end{pmatrix}=\begin{pmatrix}-q^{-1}\tilde af_2-(1-q^{-2})\tilde bf_1&-\tilde bf_2\\ -q^{-1}\tilde cf_2-(1-q^{-2})\tilde df_1&-\tilde df_2\end{pmatrix},$$

$$\Delta e_1=e_1\otimes 1+\tilde a\otimes e_1+\tilde b\otimes e_2+(1-q^{-2})(q^{-1}\tilde bf_1-\tilde af_2)\otimes e_1e_2,$$

$$\Delta e_2=e_2\otimes 1+\tilde c\otimes e_1+\tilde d\otimes e_2+(1-q^{-2})(q^{-1}\tilde df_1-\tilde cf_2)\otimes e_1e_2,$$

$$\Delta f_1=1\otimes f_1+f_1\otimes\tilde a+f_2\otimes\tilde c+(1-q^2)f_1f_2\otimes(e_1\tilde c-q^{-1}e_2\tilde a),$$

$$\Delta f_2=1\otimes f_2+f_1\otimes\tilde b+f_2\otimes\tilde d+(1-q^2)f_1f_2\otimes(e_1\tilde d-q^{-1}e_2\tilde b),$$

$$\Delta\tilde a=\tilde a\otimes\tilde a+\tilde b\otimes\tilde c+(q-q^{-1})(\tilde bf_1-q\tilde af_2)\otimes(e_1\tilde c-q^{-1}e_2\tilde a),$$

$$\Delta\tilde b=\tilde a\otimes\tilde b+\tilde b\otimes\tilde d+(q-q^{-1})(\tilde bf_1-q\tilde af_2)\otimes(e_1\tilde d-q^{-1}e_2\tilde b),$$

$$\Delta\tilde c=\tilde c\otimes\tilde a+\tilde d\otimes\tilde c+(q-q^{-1})(\tilde df_1-q\tilde cf_2)\otimes(e_1\tilde c-q^{-1}e_2\tilde a),$$

$$\Delta\tilde{d} = \tilde{c}\otimes\tilde{b} + \tilde{d}\otimes\tilde{d} + (q-q^{-1})(\tilde{d}f_1 - q\tilde{c}f_2)\otimes(e_1\tilde{d} - q^{-1}e_2\tilde{b}).$$

*Proof.* First note that

$$\mathcal{R}(S\tilde{a},\tilde{a}) = \mathcal{R}(S\tilde{d},\tilde{d}) = -1, \quad \mathcal{R}(S\tilde{a},\tilde{d}) = \mathcal{R}(S\tilde{d},\tilde{a}) = -q, \quad \mathcal{R}(S\tilde{b},\tilde{c}) = -(1-q^2)$$

and zero on other cases of this form. Then the inverse braiding is

$$\Psi^{-1}(e_1\otimes e_2) = \mathcal{R}(Se_1^{\overline{(1)}}, e_2^{\overline{(1)}})e_2^{\overline{(\infty)}}\otimes e_1^{\overline{(\infty)}} = -qe_2\otimes e_1 - (1-q^2)e_1\otimes e_2,$$

$$\Psi^{-1}(e_2\otimes e_1) = \mathcal{R}(Se_2^{\overline{(1)}}, e_1^{\overline{(1)}})e_1^{\overline{(\infty)}}\otimes e_2^{\overline{(\infty)}} = -qe_1\otimes e_2,$$

with the result that $\overline{S}(e_1 \cdot_{\underline{\mathrm{op}}} e_2) = q^2 e_1 \cdot_{\underline{\mathrm{op}}} e_2$ and $e_2 \cdot_{\underline{\mathrm{op}}} e_1 + q^{-1}e_1 \cdot_{\underline{\mathrm{op}}} e_2 = 0$ in $B^{\underline{\mathrm{op}}}$. We now apply the codouble bosonisation theorem. It is easy to see that $f_ie_j \equiv (1\otimes 1\otimes f_i)(e_j\otimes 1\otimes 1) = e_j\otimes 1\otimes 1 \equiv e_jf_i$. Next, we compute that for any $s^i{}_j \in \mathbb{C}_q[GL_2]$, where $s^1{}_1 = \tilde{a}, s^1{}_2 = \tilde{b}, s^2{}_1 = \tilde{c}, s^2{}_2 = \tilde{d}$,

$$s^i{}_je_k = e_k^{\overline{(\infty)}}(s^i{}_j)_{(2)}\mathcal{R}(S(s^i{}_j)_{(1)}, e_k^{\overline{(1)}}) = \sum_{l=1}^{2} e_k^{\overline{(\infty)}} s^l{}_j\mathcal{R}(Ss^i{}_l, e_k^{\overline{(1)}}),$$

$$f_ks^i{}_j = (s^i{}_j)_{(1)}f_k^{\overline{(0)}}\mathcal{R}(f_k^{\overline{(1)}}, (s^i{}_j)_{(2)}) = \sum_{l=1}^{2} s^i{}_l f_k^{\overline{(0)}}\mathcal{R}(f_k^{\overline{(1)}}, s^l{}_j),$$

which comes out as the stated cross relations. Now let

$$\{e_a\} = \{1, e_1, e_2, e_1e_2\}, \quad \{f^a\} = \{1, f_1, f_2, f_1f_2\}$$

be a basis and dual basis of $B, B^*$ respectively. Then

$$\Delta e_i = e_i\otimes 1 + \sum_{a=1}^{2} e_i^{\overline{(1)}}{}_{(1)}f^a\otimes(e_{a\underline{(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} e_i^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(2)}})\mathcal{R}(e_{a\underline{(1)}}{}^{\overline{(1)}}, e_i^{\overline{(1)}}{}_{(2)}),$$

$$\Delta f_i = 1\otimes f_i + \sum_{a=1}^{2} f^a\otimes(e_{a\underline{(1)}} \cdot_{\underline{\mathrm{op}}} \overline{S}e_{a\underline{(3)}}{}^{\overline{(\infty)}})f_i^{\overline{(1)}}{}_{(2)}\mathcal{R}(Sf_i^{\overline{(1)}}{}_{(1)}, e_{a\underline{(3)}}{}^{\overline{(\infty)}})\langle f_i, e_{a\underline{(2)}}\rangle,$$

$$\Delta s^i{}_j = \sum_{a,k,l,r=1}^{2} s^i{}_k f^a \otimes (e_{\underline{a(1)}}{}^{\overline{(\infty)}} \cdot_{\underline{\mathrm{op}}} \overline{S} e_{\underline{a(2)}}{}^{\overline{(\infty)}}) s^r{}_j \mathcal{R}(e_{\underline{a(1)}}{}^{\overline{(1)}}, s^k{}_l) \mathcal{R}(S s^l{}_r, e_{\underline{a(2)}}{}^{\overline{(1)}}),$$

which come out as stated for all $i, j \in \{1, 2\}$. Finally, we let

$$s_1 = (q - q^{-1})(\tilde{b} f_1 - q\tilde{a} f_2), \quad s_2 = (q - q^{-1})(\tilde{d} f_1 - q\tilde{c} f_2),$$

$$t_1 = e_1\tilde{c} - q^{-1} e_2 \tilde{a}, \quad t_2 = e_1 \tilde{d} - q^{-1} e_2 \tilde{b},$$

and write $\mathbb{C}_q^{fer}[SL_3]$ as having a matrix of generators $t^i{}_j$, where now $i, j \in \{1, 2, 3\}$, by

$$\mathbf{t} = \begin{pmatrix} t^1{}_1 & t^1{}_2 & t^1{}_3 \\ t^2{}_1 & t^2{}_2 & t^2{}_3 \\ t^3{}_1 & t^3{}_2 & t^3{}_3 \end{pmatrix} = \begin{pmatrix} x & t_1 & t_2 \\ s_1 & \tilde{a} & \tilde{b} \\ s_2 & \tilde{c} & \tilde{d} \end{pmatrix};$$

$$x = D + (t_1 D^{-1}(\tilde{d} s_1 - q\tilde{b} s_2) - q^{-1} t_2 D^{-1}(\tilde{c} s_1 - q\tilde{a} s_2)). \tag{4.3.1}$$

Here $D$ obeys $Dt_i = qt_iD$ and $Ds_i = qs_iD$ for $i = 1, 2$. The coproduct now has the standard matrix form $\Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}$ and in these terms the quadratic relations are

$$(t^1{}_2)^2 = (t^1{}_3)^2 = (t^2{}_1)^2 = (t^3{}_1)^2 = 0,$$

$$[t^1{}_2, t^1{}_1]_{q^{-1}} = [t^1{}_3, t^1{}_1]_{q^{-1}} = [t^2{}_1, t^1{}_1]_{q^{-1}} = [t^3{}_1, t^1{}_1]_{q^{-1}} = [t^2{}_3, t^2{}_2]_q = [t^3{}_2, t^2{}_2]_q = 0,$$

$$[t^3{}_3, t^2{}_3]_q = [t^3{}_3, t^3{}_2]_q = [t^2{}_1, t^1{}_2] = [t^2{}_1, t^1{}_3] = [t^3{}_1, t^1{}_2] = [t^3{}_1, t^1{}_3] = [t^3{}_2, t^2{}_3] = 0,$$

$$[t^2{}_2, t^1{}_1] = -\lambda t^1{}_2 t^2{}_1, \quad [t^2{}_3, t^1{}_1] = -\lambda t^1{}_3 t^2{}_1, \quad [t^3{}_2, t^1{}_1] = -\lambda t^1{}_2 t^3{}_1,$$

$$[t^3{}_3, t^1{}_1] = -\lambda t^1{}_3 t^3{}_1, \quad [t^3{}_3, t^2{}_2] = \lambda t^2{}_3 t^3{}_2, \quad \{t^1{}_3, t^1{}_2\}_{q^{-1}} = \{t^3{}_1, t^2{}_1\}_{q^{-1}} = 0,$$

$$\{t^2{}_2, t^1{}_2\}_q = \{t^2{}_2, t^2{}_1\}_q = \{t^2{}_3, t^1{}_3\}_q = \{t^2{}_3, t^2{}_1\}_q = \{t^3{}_2, t^1{}_2\}_q = \{t^3{}_2, t^3{}_1\}_q = 0,$$

$$\{t^3{}_3, t^1{}_3\}_q = \{t^3{}_3, t^3{}_1\}_q = \{t^2{}_2, t^1{}_3\} = \{t^3{}_1, t^2{}_2\} = \{t^3{}_1, t^2{}_3\} = \{t^3{}_2, t^1{}_3\} = 0,$$

$$\{t^2{}_3, t^1{}_2\} = -\lambda t^1{}_3 t^2{}_2, \quad \{t^3{}_3, t^1{}_2\} = -\lambda t^1{}_3 t^3{}_2, \quad \{t^3{}_2, t^2{}_1\} = \lambda t^2{}_2 t^3{}_1, \quad \{t^3{}_3, t^2{}_1\} = \lambda t^2{}_3 t^3{}_1,$$

where $[\ ,\ ]_q$ is as before, similarly $\{a,b\}_q = ab+qba$ for any $a,b$ is the $q$-anti-commutator, and $\lambda = q-q^{-1}$. Using Lemma 3.2.7, the values $\mathcal{R}(t^i{}_j, t^k{}_l)$ of the coquasitriangular structure of $\mathbb{C}_q^{fer}[SL_3]$ come out, in the same conventions as in the proof of part (2) of Theorem 4.2.1, as

$$R^I{}_J = \begin{pmatrix} q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & -q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 \\ 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 & -q^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that since $t^1{}_1$ was defined in terms of the other generators including the $q$-sub-determinant $D = t^2{}_2t^3{}_3 - q^{-1}t^2{}_3t^3{}_2$, there are in fact only 8 algebra generators and 28 $q$-(anti)commutation relations other than the nilpotency ones and those involving $t^1{}_1$, putting this conceptually on a par with $\mathbb{C}_q[SL_3]$. Instead of a cubic $q$-determinant relation, we can regard (4.3.1) as the cubic-quartic relation

$$Dt^1{}_1 - qt^1{}_2(t^3{}_3t^2{}_1 - qt^2{}_3t^3{}_1) + t^1{}_3(t^3{}_2t^2{}_1 - qt^2{}_2t^3{}_1) = D^2.$$

Also note that (2.2.5) in the ‘R-matrix’ form $R^i{}_m{}^k{}_n t^m{}_j t^n{}_l = t^k{}_n t^i{}_m R^m{}_j{}^n{}_l$ (sum over repeated indices) encodes exactly the quadratic relations above for $\mathbb{C}_q^{fer}[SL_3]$ including the nilpotent ones. $\square$

## 4.4 Appendix

Here we give the detailed calculation for the formulae in Theorem 4.2.1 since they are complicated and were omitted there.

**Lemma 4.4.1.** *The Formula of $\Delta x_1$ and $\Delta x_2$ is as stated as in Theorem 4.2.1.*

*Proof.* Write $\Delta x_1 = x_1 \otimes 1 + A_1 + A_2$, and divide the calculation into two parts. We suppress the index of summation, and remember that in the summation, $r, s$ run from 0 to $n-1$, $r_1$ runs from 0 to $r$, and $s_1$ runs from 0 to $s$. First, we have

$$\begin{aligned}
A_1 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+\frac{(r_1+s_1)(r_1+s_1+1)}{2}+r_1+s_1} \frac{\tilde{a} y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^{r+1} x_2^s \\
=& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{rs_1+\frac{r_1(r_1+1)}{2}+\frac{s_1(s_1+1)}{2}} (-1)^{r+s-r_1-s_1} \frac{\tilde{a} y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^{r+1} x_2^s \\
=& \sum (q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{a} y_1^r y_2^s \otimes x_1^{r+1} x_2^s,
\end{aligned}$$

where we use (4.2.3) in the last equality. For the second term, notice that $1+[r_1]_q(q-1) = q^{r_1}$. Thus, we have

$$\begin{aligned}
A_2 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}+s_1-mr+r_1} \frac{\tilde{b} y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^r x_2^{s+1} \\
=& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{-mr+rs_1+\frac{r_1(r_1+1)}{2}+\frac{s_1(s_1+1)}{2}} \tilde{b} y_1^r y_2^s \otimes x_1^r x_2^{s+1} \\
=& \sum q^{-mr} (q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{b} y_1^r y_2^s \otimes x_1^r x_2^{s+1},
\end{aligned}$$

where we use (4.2.3) in the last equality. Thus we obtained $\Delta x_1$ as stated in Theorem 4.2.1. The calculation for $\Delta x_2$ is similar. $\square$

**Lemma 4.4.2.** *The formula of $\Delta y_1$ and $\Delta y_2$ is as stated in Theorem 4.2.1.*

*Proof.* Write $\Delta y_1 = 1 \otimes y_1 + B_1 + B_2$, and divide the calculation into two parts. First, we have

$$\begin{aligned}
B_1 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [r_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+\frac{(r_1+s_1-1)(r_1+s_1-2)}{2}-r+ms+r_1+s_1} \\
& \times \frac{y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^{r-1} x_2^s \tilde{a} \\
=& \sum q^{-r+ms+1} \Big( \frac{(-1)^{r-r_1} q^{\frac{r_1(r_1+1)}{2}} q^{-r_1}[r_1]_q}{[r_1]_q![r-r_1]_q!} \Big) \Big( \frac{(-1)^{s-s_1} q^{\frac{s_1(s_1+1)}{2}} q^{s_1(r-1)}}{[s_1]_q![s-s_1]_q!} \Big) \\
& \times y_1^r y_2^s \otimes x_1^{r-1} x_2^s \tilde{a} \\
=& \sum q^{-r+ms+1} (q-1)^s \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q \Big( \frac{-(q-1)^r}{(1-q)} \Big) y_1^r y_2^s \otimes x_1^{r-1} x_2^s \tilde{a} \\
=& \sum \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q (q-1)^{r+s-1} q^{-r+ms+1} y_1^r y_2^s \otimes x_1^{r-1} x_2^s \tilde{a},
\end{aligned}$$

where we use (4.2.3) in the third equality. For $B_2$, one can find that

$$[s_1]_q + [r_1]_q [s-s_1]_q = [s]_q - \frac{q^{r_1+s_1}}{1-q} + \frac{q^{r_1+s}}{1-q}.$$

Thus, we have the following

$$\begin{aligned}
B_2 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{r_1+s_1-r-s+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} (-1)^{r+s-r_1-s_1} ([s]_q - \frac{q^{r_1+s_1}}{1-q} + \frac{q^{r_1+s}}{1-q}) \\
& \times \frac{y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^r x_2^{s-1} \tilde{c} \\
=& \sum \frac{(-1)^{r+s-r_1-s_1} q^{-r-s+1-r_1+s_1(r-1)+\frac{r_1(r_1+1)}{2}+\frac{s_1(s_1+1)}{2}}}{[r_1]_q![r-r_1]_q![s_1]_q![s-s_1]_q!} ([s]_q - \frac{q^{r_1+s_1}}{1-q} + \frac{q^{r_1+s}}{1-q}) \\
& \times y_1^r y_2^s \otimes x_1^r x_2^{s-1} \tilde{c}.
\end{aligned}$$

We write $B_2 = (b_1 + b_2 + b_3) y_1^r y_2^s \otimes x_1^r x_2^{s-1} \tilde{c}$, where $b_1, b_2, b_3$ are the products of the fraction preceding the bracket with $[s]_q$, $\frac{q^{r_1+s_1}}{1-q}$, and $\frac{q^{r_1+s}}{1-q}$ respectively. There will be no contribution from $b_1$ since $b_1 = 0$. By using (4.2.3), we have

$$\begin{aligned} b_2 =& \sum q^{-r-s+1}\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}}{[r_1]_q![r-r_1]_q!(q-1)}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{rs_1}}{[s_1]_q![s-s_1]_q!}\Big) \\ =& \sum q^{-r-s+1}(q-1)^{r+s-1}\begin{bmatrix} r+s \\ s \end{bmatrix}_q . \end{aligned}$$

Similarly, by using (4.2.3), we have

$$\begin{aligned} b_3 =& \sum q^{-r-s+1}\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{s_1(r-1)+s}}{[s_1]_q![s-s_1]_q!}\Big) \\ =& -\sum q^{-r-s+1}(q-1)^{r+s-1}\begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q q^s. \end{aligned}$$

Combine them together, we have

$$B_2 = \sum q^{-r-s+1}(q-1)^{r+s-1}\Big(\begin{bmatrix} r+s \\ s \end{bmatrix}_q - \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q q^s\Big) y_1^r y_2^s \otimes x_1^r x_2^{s-1}\tilde{c}.$$

A bit more calculation gives us that

$$\begin{aligned} \Big(\begin{bmatrix} r+s \\ s \end{bmatrix}_q - \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q q^s\Big) =& \frac{[r+s-1]_q!}{[r-1]_q![s]_q!}\Big(\frac{[r+s]_q}{[r]_q} - q^s\Big) \\ =& \frac{[r+s-1]_q!}{[r-1]_q![s]_q!}.\frac{[s]_q}{[r]_q} = \frac{[r+s-1]_q!}{[r]_q![s-1]_q!} = \begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q , \end{aligned}$$

which simplifies $B_2$ into

$$B_2 = \sum q^{-r-s+1}(q-1)^{r+s-1}\begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q y_1^r y_2^s \otimes x_1^r x_2^{s-1}\tilde{c}.$$

Thus we obtain the formula of $\Delta y_1$ as stated in Theorem 4.2.1. Similarly for $\Delta y_2$. $\square$

**Lemma 4.4.3.** *The formula of $\Delta\varsigma$ is as stated in Theorem 4.2.1.*

*Proof.* We compute directly that

$$\begin{aligned}
\Delta\varsigma =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{s_1(r-r1)+m\beta(2r_1+2s_1-r-s)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}}(-1)^{r+s-r_1-s_1} \\
&\times \frac{\varsigma y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^r x_2^s \varsigma \\
=& \sum q^{-m\beta(r+s)}\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}q^{(2m\beta-1)r_1}}{[r_1]_q![s-s_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{2m\beta+r-1}s_1}{[s_1]_q![s-s_1]_q!}\Big) \\
&\times \varsigma y_1^r y_2^s \otimes x_1^r x_2^s \varsigma \\
=& \sum q^{-m\beta(r+s)}(q-1)^{r+s} \begin{bmatrix} r+2m\beta-1 \\ r \end{bmatrix}_q \begin{bmatrix} r+s+2m\beta-1 \\ s \end{bmatrix}_q \varsigma y_1^r y_2^s \otimes x_1^r x_2^s \varsigma,
\end{aligned}$$

where the last equality is obtained by (4.2.3). □

**Lemma 4.4.4.** *The formula of $\Delta\tilde{a}, \Delta\tilde{b}, \Delta\tilde{c}$ and $\Delta\tilde{d}$ are as stated in Theorem 4.2.1.*

*Proof.* Write the formula of $\Delta\tilde{a}$ in the proof of Theorem 4.2.1 into four parts, i.e. $\Delta\tilde{a} = a_1 + a_2 + a_3 + a_4$. First, we compute that

$$\begin{aligned}
a_1 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+2r_1+s_1-r+ms+\frac{(r_1+s1)(r_1+s_1-1)}{2}} \\
&\times \frac{\tilde{a} y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^r x_2^s \tilde{a} \\
=& \sum q^{-r+ms}\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}q^{r_1}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{rs_1}}{[s_1]_q![s-s_1]_q!}\Big)\tilde{a} y_1^r y_2^s \otimes x_1^r x_2^s \tilde{a} \\
=& \sum q^{-r+ms}(q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q [r+1]_q \tilde{a} y_1^r y_2^s \otimes x_1^r x_2^s \tilde{a},
\end{aligned}$$

where we use (4.2.3) in the last equality. Next, we compute that

$$
\begin{aligned}
a_2 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [s-s_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+2r_1+2s_1-r-s+\frac{(r_1+s1)(r_1+s_1-1)}{2}} \\
& \times (1-q)\frac{\tilde{a}y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^{r+1}x_2^{s-1}\tilde{c} \\
=& \sum q^{-r-s}(1-q)\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}q^{r_1}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{(r+1)s_1}[s-s_1]_q}{[s_1]_q![s-s_1]_q!}\Big) \\
& \times \tilde{a}y_1^r y_2^s \otimes x_1^{r+1}x_2^{s-1}\tilde{c} \\
=& q^{-r-s}(q-1)^{r+s}[r+1]_q\Big(\begin{bmatrix} r+s+1 \\ s \end{bmatrix}_q - q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q\Big)\tilde{a}y_1^r y_2^s \otimes x_1^{r+1}x_2^{s-1}\tilde{c} \\
=& \sum q^{-r-s}(q-1)^{r+s}\begin{bmatrix} r+s \\ s \end{bmatrix}_q [s]_q \tilde{a}y_1^r y_2^s \otimes x_1^{r+1}x_2^{s-1}\tilde{c}.
\end{aligned}
$$

Where we simplify the last bracket in the second equality using (4.2.3) into

$$
\frac{(q-1)^s}{1-q}\Big(\begin{bmatrix} r+s+1 \\ s \end{bmatrix}_q - q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q\Big).
$$

and further calculation again by (4.2.3) gives the third equality. Next, we have

$$
\begin{aligned}
a_3 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [r_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+r_1+s_1+mr+ms+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} \\
& \times (q^{-m}-q^m)\frac{\tilde{b}y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^{r-1}x_2^{s+1}\tilde{a} \\
=& \sum q^{m(r+s)}\Big(\frac{(-1)^{r-r_1}q^{\frac{r_1(r_1+1)}{2}}[r_1]_q(q^{-m}-q^m)}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1}q^{\frac{s_1(s_1+1)}{2}}q^{rs_1}}{[s_1]_q![s-s_1]_q!}\Big) \\
& \times \tilde{b}y_1^r y_2^s \otimes x_1^{r-1}x_2^{s+1}\tilde{a} \\
=& \sum q^{m(r+s-1)}(q-1)^{r+s}\begin{bmatrix} r+s \\ s \end{bmatrix}_q [r]_q \tilde{b}y_1^r y_2^s \otimes x_1^{r-1}x_2^{s+1}\tilde{a},
\end{aligned}
$$

where the first bracket in the second equality is simplified to $q^{-m}(q-1)^r[r]_q$, and we therefore we obtain the third equality by (4.2.3). Finally, we have

$$
\begin{aligned}
a_4 =& \sum \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+\frac{(r_1+s1)(r_1+s_1-1)}{2}+r_1+2s_1+mr-s} \\
& \times (1-[r_1]_q[s-s_1]_q(q-1)^2)\frac{\tilde{b}y_1^r y_2^s}{[r]_q![s]_q!} \otimes x_1^r x_2^s \tilde{c}. \\
=& \sum \frac{(-1)^{r+s-r_1-s_1} q^{mr-s} q^{\frac{r_1(r_1+1)+s_1(s_1+1)}{2}+(r+1)s_1}}{[r_1]_q![r-r_1]_q![s_1]_q![s-s_1]_q!}(1-[r_1]_q[s-s_1]_q(q-1)^2) \\
& \times \tilde{b}y_1^r y_2^s \otimes x_1^r x_2^s \tilde{c}.
\end{aligned}
$$

Notice that

$$
1-[r_1]_q[s-s_1]_q(q-1)^2 = q^{r_1} + q^{s-s_1} - q^{r_1-s_1+s}
$$

Thus we write $a_4 = (b_1+b_2-b_3)\tilde{b}y_1^r y_2^s \otimes x_1^r x_2^s \tilde{c}$, where $b_1, b_2, b_3$ are the products of the fraction preceding the bracket with $q^{r_1}, q^{s-s_1}, q^{r_1-s_1+s}$ respectively. By using (4.2.3), we have

$$
\begin{aligned}
b_1 =& \sum q^{mr-s}\Big(\frac{(-1)^{r-r_1} q^{\frac{r_1(r_1+1)}{2}} q^{r_1}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1} q^{\frac{s_1(s_1+1)}{2}} q^{(r+1)s_1}}{[s_1]_q![s-s_1]_q!}\Big) \\
=& \sum q^{mr-s}(q-1)^{r+s} \begin{bmatrix} r+s+1 \\ s \end{bmatrix}_q [r+1]_q. \\
b_2 =& \sum q^{mr-s}\Big(\frac{(-1)^{r-r_1} q^{\frac{r_1(r_1+1)}{2}}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1} q^{\frac{s_1(s_1+1)}{2}} q^{rs_1+s}}{[s_1]_q![s-s_1]_q!}\Big) \\
=& \sum q^{mr-s}(q-1)^{r+s} q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q. \\
b_3 =& \sum q^{mr-s}\Big(\frac{(-1)^{r-r_1} q^{\frac{r_1(r_1+1)}{2}} q^{r_1}}{[r_1]_q![r-r_1]_q!}\Big)\Big(\frac{(-1)^{s-s_1} q^{\frac{s_1(s_1+1)}{2}} q^{rs_1+s}}{[s_1]_q![s-s_1]_q!}\Big) \\
=& \sum q^{mr-s}(q-1)^{r+s} q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q [r+1]_q.
\end{aligned}
$$

By combining them, we have

$$
\begin{aligned}
a_4 =& \sum q^{mr-s}(q-1)^{r+s}\Big( \begin{bmatrix} r+s+1 \\ s \end{bmatrix}_q [r+1]_q + q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q - q^s \begin{bmatrix} r+s \\ s \end{bmatrix}_q [r+1]_q \Big) \\
& \times \tilde{b} y_1^r y_2^s \otimes x_1^r x_2^s \tilde{c} \\
=& \sum q^{mr-s}(q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q [s+1]_q \tilde{b} y_1^r y_2^s \otimes x_1^r x_2^s \tilde{c}.
\end{aligned}
$$

Therefore $\Delta\tilde{a}$ is as stated in Theorem 4.2.1. Similarly for $\Delta\tilde{b}, \Delta\tilde{c}$ and $\Delta\tilde{d}$. □

# Chapter 5

# Exterior algebra on double cross (co)product Hopf algebras

We will first recall the concept of *differentiable coaction* in Section 5.1, where the coaction $B \to B \otimes A$ extends to $\Omega(B) \to \Omega(B)\underline{\otimes}\Omega(A)$ as a map of exterior algebra, making $\Omega(B)$ a super $\Omega(A)$-comodule algebra, and also introduce the concept of *differentiable action*, where the action $\triangleleft : B \otimes A \to B$ of module algebra extends to $\Omega(B)\underline{\otimes}\Omega(A) \to \Omega(B)$, making $\Omega(B)$ a super $\Omega(A)$-module algebra.

Then in Section 5.2, for bialgebras or Hopf algebras $A, H$ forming a double cross product $A \bowtie H$, we construct a super double cross product $\Omega(A) \bowtie \Omega(H)$ and show that it gives a strongly bicovariant exterior algebra on the double cross product $A \bowtie H$. As a special case, for dually paired Hopf algebras $A, H$, the differential calculus on the quantum double $D(A,H) = A^{\mathrm{op}} \bowtie H$ is discussed in Section 5.2.1. Moreover, we show that $\Omega(D(A,H))$ acts on $H$ differentiably. Similarly, differential calculus on the double cross product of coquasitriangular Hopf algebras $A \bowtie_{\mathcal{R}} A$ is discussed in Section 5.2.2 and we show that $\Omega(A) \bowtie_{\mathcal{R}} \Omega(A)$ acts and coacts on $A$ differentiably. For completeness, we also give a construction of differentials on the double cross coproduct $H \blacktriangleright\!\!\blacktriangleleft A$ in Section 5.3 such that it coacts on $H$ and $A$ differentiably.

## 5.1 Differentiable coactions and actions

**Definition 5.1.1.** Let $B$ be a right $A$-comodule algebra with coaction $\Delta_R b = b^{\overline{(0)}} \otimes b^{\overline{(1)}}$ equipped with an exterior differential algebra $\Omega(B)$ which is a right $A$-comodule algebra with right coaction $\Delta_R \eta = \eta^{\overline{(0)}} \otimes \eta^{\overline{(1)}}$ on each degree such that d is a comodule map. The coaction $\Delta_R$ is called *differentiable* if it extends to a degree-preserving map $\Delta_{R*} : \Omega(B) \to \Omega(B)\underline{\otimes}\Omega(A)$ of differential exterior algebras, where we take the graded tensor product, and $\Delta_{R*}$ respects d of $\Omega(B)\underline{\otimes}\Omega(A)$ in the sense that $\Delta_{R*}\mathrm{d}_B = \mathrm{d}\Delta_{R*}$, or explicitly

$$\Delta_{R*}\mathrm{d}_B\eta = \mathrm{d}_B\eta^{\overline{(0)}*} \otimes \eta^{\overline{(1)}*} + (-1)^{|\eta|}\eta^{\overline{(0)}*} \otimes \mathrm{d}_A\eta^{\overline{(1)}*}, \tag{5.1.1}$$

where we denote $\Delta_{R*}\eta = \eta^{\overline{(0)}*} \otimes \eta^{\overline{(1)}*} \in \Omega(B)\underline{\otimes}\Omega(A)$ for all $\eta \in \Omega(B)$.

If $\Delta_{R*}$ exists then it is uniquely determined from $\Delta_R$. For instance on $\Omega^1(B)$, we would need

$$\Delta_{R*}(b\mathrm{d}_B c) = b^{\overline{(0)}}\mathrm{d}_B c^{\overline{(0)}} \otimes b^{\overline{(1)}}c^{\overline{(1)}} + b^{\overline{(0)}}c^{\overline{(0)}} \otimes b^{\overline{(1)}}\mathrm{d}_A c^{\overline{(1)}}. \tag{5.1.2}$$

where the first term is the coaction $\Delta_R : \Omega^1(B) \to \Omega^1(B) \otimes A$ and the second term is map

$$\delta_R(b\mathrm{d}_B c) = b^{\overline{(0)}}c^{\overline{(0)}} \otimes b^{\overline{(1)}}\mathrm{d}_A c^{\overline{(1)}}, \quad \delta_R : \Omega^1(B) \to B \otimes \Omega^1(A).$$

which we require to be well-defined.

**Lemma 5.1.2.** *[7] For $\Omega(B)$ an $A$-covariant calculus, if the map $\Delta_{R*} : \Omega^1(B) \to (\Omega^1(B) \otimes A) \oplus (B \otimes \Omega^1(A))$ in (5.1.2) is well-defined and $\Omega(B)$ is the maximal prolongation of $\Omega^1(B)$ then $\Delta_{R*}$ extends to all degrees and the coaction $\Delta_R$ is differentiable.*

*Proof.* This is in [7] but to be self-contained, we give our own short proof. In fact, it suffices to prove that $\Delta_{R*}$ extends to $\Omega^2(B)$ since the maximal prolongation is quadratic. So we need to check that $\Delta_{R*}(\xi\eta) = \Delta_{R*}(\xi)\Delta_{R*}(\eta)$ is well-defined for $\xi, \eta \in \Omega^1(B)$. Suppose we have the relation $b\mathrm{d}_B c = 0$ in $\Omega^1(B)$ (sum of such terms understood) which implies that $\mathrm{d}_B b \mathrm{d}_B c = 0 \in \Omega^2(B)$. Applying $\Delta_{R*}$ to the relation in $\Omega^1(B)$ we have

$$b^{\overline{(0)}}\mathrm{d}_B c^{\overline{(0)}} \otimes b^{\overline{(1)}} c^{\overline{(1)}} = 0, \quad b^{\overline{(0)}} c^{\overline{(0)}} \otimes b^{\overline{(1)}}\mathrm{d}_A c^{\overline{(1)}} = 0.$$

Applying $\mathrm{id}\otimes\mathrm{d}_A$ to the first equation and $\mathrm{d}_B\otimes\mathrm{id}$ to the second equation then subtracting them gives us

$$b^{\overline{(0)}}\mathrm{d}_B c^{\overline{(0)}} \otimes (\mathrm{d}_A b^{\overline{(1)}}) c^{\overline{(1)}} - (\mathrm{d}_B b^{\overline{(0)}}) c^{\overline{(0)}} \otimes b^{\overline{(1)}}\mathrm{d}_A c^{\overline{(1)}} = 0,$$

which is the $\Omega^1(B)\otimes\Omega^1(A)$ part of $\Delta_{R*}(\mathrm{d}_B b \mathrm{d}_B c)$. Applying $\mathrm{d}_B\otimes\mathrm{id}$ to the first equation gives

$$\mathrm{d}_B b^{\overline{(0)}}\mathrm{d}_B c^{\overline{(0)}} \otimes b^{\overline{(1)}} c^{\overline{(1)}} = 0$$

which is the $\Omega^2(B)\otimes\mathrm{id}$ part. Finally, applying $\mathrm{id}\otimes\mathrm{d}_A$ to the second equation gives

$$b^{\overline{(0)}} c^{\overline{(0)}} \otimes \mathrm{d}_A b^{\overline{(1)}}\mathrm{d}_A c^{\overline{(1)}} = 0$$

which is the $B\otimes\Omega^2(A)$ part. Since all relations in the maximal prolongation are sent to zero then $\Delta_{R*}$ extends to $\Omega^2(B)$, which completes the proof. □

There is an equally good left-handed definition of differentiable coaction, where the left coaction $\Delta_L : B \to A\otimes B$ extends to a degree-preserving map $\Delta_{L*} : \Omega(B) \to \Omega(A)\underline{\otimes}\Omega(B)$ of exterior algebra. We also need a concept of differentiable action, which we introduce now

**Definition 5.1.3.** Let $B$ be a right $A$-module algebra with action $\triangleleft : B\otimes A \to A$, equipped with an $A$-covariant exterior algebra $\Omega(B)$, i.e., $\Omega(B)$ is also a right $A$-module algebra with action $\triangleleft$ and d an $A$-module map. This action is called *differentiable* if it extends to a degree-preserving map $\triangleleft : \Omega(B)\underline{\otimes}\Omega(A) \to \Omega(B)$, making $\Omega(B)$ a super $\Omega(A)$-module algebra, and $\triangleleft$ respects d of $\Omega(B)\underline{\otimes}\Omega(A)$ in the sense that $\mathrm{d}_B\triangleleft = \triangleleft\mathrm{d}$, or explicitly

$$\mathrm{d}_B(\eta\triangleleft\omega) = (\mathrm{d}_B\eta)\triangleleft\omega + (-1)^{|\eta|}\eta\triangleleft(\mathrm{d}_A\omega). \tag{5.1.3}$$

If the action is differentiable, then it is uniquely determined. For instance, on $\Omega^1(B)$ and $\Omega^2(B)$, we have

$$\mathrm{d}_B(b\triangleleft a) = (\mathrm{d}_B b)\triangleleft a + b\triangleleft\mathrm{d}_A a, \quad \mathrm{d}_B((\mathrm{d}_B b)\triangleleft a) = -(\mathrm{d}_B b)\triangleleft\mathrm{d}_A a$$

where the $(\mathrm{d}_B b)\triangleleft a$ is given as is $\mathrm{d}_B(b\triangleleft a)$, hence $b\triangleleft\mathrm{d}_A a$ is determined, and hence also $\triangleleft : B \otimes \Omega^1(A) \to \Omega^1(B)$. Similarly the second equation specified $\triangleleft : \Omega^1(B)\underline{\otimes}\Omega^1(A) \to \Omega^2(B)$. The action also obeys module algebra axiom, for example

$$(\eta\xi)\triangleleft\mathrm{d}_A a = (\eta\triangleleft a_{(1)})(\xi\triangleleft\mathrm{d}_A a_{(2)}) + (-1)^{|\xi|}(\eta\triangleleft\mathrm{d}_A a_{(1)})(\xi\triangleleft a_{(2)}),$$

for all $\eta, \xi \in \Omega(B)$ and $a \in A$, hence $\Omega(B)\underline{\otimes}\Omega^1(A) \to \Omega(B)$ is specified.

**Lemma 5.1.4.** *Let $B$ be an $A$-module algebra with an exterior algebra $\Omega(B)$ which is also an $A$-module algebra and let $\Omega(A)$ be the maximal prolongation of $\Omega^1(A)$. If $\triangleleft : \Omega(B)\underline{\otimes}\Omega^1(A) \to \Omega(B)$ is well-defined by*

$$\eta\triangleleft((\mathrm{d}_A a)c) := (\eta\triangleleft\mathrm{d}_A a)\triangleleft c, \quad \eta\triangleleft\mathrm{d}_A a := (-1)^{|\eta|}\big(\mathrm{d}_B(\eta\triangleleft a) - (\mathrm{d}_B\eta)\triangleleft a\big)$$

*for all $\eta \in \Omega(A)$ and $a, c \in A$ then it extends to $\triangleleft : \Omega(B)\underline{\otimes}\Omega(A) \to \Omega(B)$ as a differentiable action.*

*Proof.* First we check that $\triangleleft : \Omega(B)\underline{\otimes}\Omega^1(A) \to \Omega(B)$ if defined as above gives an action of $\Omega^1(A)$ in the sense

$$\begin{aligned}\eta\triangleleft(a\mathrm{d}_A c) =& \eta\triangleleft(\mathrm{d}_A(ac) - (\mathrm{d}_A a)c) = (-1)^{|\eta|}\mathrm{d}_B(\eta\triangleleft ac) - (-1)^{|\eta|}(\mathrm{d}_B\eta)\triangleleft ac - (\eta\triangleleft\mathrm{d}_A a)\triangleleft c\\ =& (-1)^{|\eta|}\mathrm{d}_B((\eta\triangleleft a)\triangleleft c) - (-1)^{|\eta|}((\mathrm{d}_B\eta)\triangleleft a)\triangleleft c - (\eta\triangleleft\mathrm{d}_A a)\triangleleft c\end{aligned}$$

$$
\begin{aligned}
&=(-1)^{|\eta|}(\mathrm{d}_B(\eta\triangleleft a))\triangleleft c+(\eta\triangleleft a)\triangleleft\mathrm{d}_A c-(-1)^{|\eta|}((\mathrm{d}_B\eta)\triangleleft a)\triangleleft c-(\eta\triangleleft\mathrm{d}_A a)\triangleleft c\\
&=(-1)^{|\eta|}((\mathrm{d}_B\eta)\triangleleft a)\triangleleft c+(\eta\triangleleft\mathrm{d}_A a)\triangleleft c+(\eta\triangleleft a)\triangleleft\mathrm{d}_A c-(-1)^{|\eta|}((\mathrm{d}_B\eta)\triangleleft a)\triangleleft c-(\eta\triangleleft\mathrm{d}_A a)\triangleleft c\\
&=(\eta\triangleleft a)\triangleleft\mathrm{d}_A c.
\end{aligned}
$$

We now suppose that $\Omega(A)$ is the maximal prolongation of $\Omega^1(A)$ and show that we can extend the above to a right action of $\Omega(A)$ of all degrees. The higher relations relations are quadratic of the form $\mathrm{d}_A a\mathrm{d}_A c=0$ for $a\mathrm{d}_A c=0$ (a sum of such terms understood) and we check that

$$
0=\mathrm{d}_B(\eta\triangleleft(a\mathrm{d}_A c))=(\mathrm{d}_B\eta)\triangleleft(a\mathrm{d}_A c)+(-1)^{|\eta|}\eta\triangleleft(\mathrm{d}_A a\mathrm{d}_A c)=(-1)^{|\eta|}\eta\triangleleft(\mathrm{d}_A a\mathrm{d}_A c)
$$

as required. Indeed this gives an action of $\Omega^2(A)$ in the sense

$$
\begin{aligned}
\eta\triangleleft\mathrm{d}_A a\mathrm{d}_A c&=\eta\triangleleft\mathrm{d}_A(a\mathrm{d}_A c)=(-1)^{|\eta|}(\mathrm{d}_B(\eta\triangleleft a\mathrm{d}_A c)-(\mathrm{d}_B\eta)\triangleleft a\mathrm{d}_A c)\\
&=(-1)^{|\eta|}\big(\mathrm{d}_B((\eta\triangleleft a)\triangleleft\mathrm{d}_A c)-((\mathrm{d}_B\eta)\triangleleft a)\triangleleft\mathrm{d}_A c\big)=(\eta\triangleleft\mathrm{d}_A a)\triangleleft\mathrm{d}_A c,
\end{aligned}
$$

and since $\Omega(A)$ is the maximal prolongation of $\Omega^1(A)$, then there is no new relations in higher degree and thus $\triangleleft$ can be extended further to be an action of $\Omega(A)$ of all degrees.

Next we check that $\Omega(B)$ is a super right $\Omega(A)$-module algebra with regard to the action of $\Omega^1(A)$,

$$
\begin{aligned}
(\eta\xi)\triangleleft\mathrm{d}_A a&=(-1)^{|\eta\xi|}(\mathrm{d}_B((\eta\xi)\triangleleft a)-\mathrm{d}_B(\eta\xi)\triangleleft a)\\
&=(-1)^{|\eta|+|\xi|}\mathrm{d}_B\big((\eta\triangleleft a_{(1)})(\xi\triangleleft a_{(2)})-((\mathrm{d}_B\eta)\xi)\triangleleft a-(-1)^{|\eta|}(\eta\mathrm{d}_B\xi)\triangleleft a\big)\\
&=(-1)^{|\eta|+|\xi|}(\mathrm{d}_B(\eta\triangleleft a_{(1)}))(\xi\triangleleft a_{(2)})+(-1)^{|\xi|}(\eta\triangleleft a_{(1)})\mathrm{d}_B(\xi\triangleleft a_{(2)})-(-1)^{|\eta|+|\xi|}((\mathrm{d}_B\eta)\xi)\triangleleft a\\
&\quad-(-1)^{|\xi|}(\eta\mathrm{d}_B\xi)\triangleleft a\\
&=(-1)^{|\eta|+|\xi|}((\mathrm{d}_B\eta)\triangleleft a_{(1)})(\xi\triangleleft a_{(2)})+(-1)^{|\xi|}(\eta\triangleleft\mathrm{d}_A a_{(1)})(\xi\triangleleft a_{(2)})+(-1)^{|\xi|}(\eta\triangleleft a_{(1)})((\mathrm{d}_B\xi)\triangleleft a_{(2)})\\
&\quad+(\eta\triangleleft a_{(1)})(\xi\triangleleft\mathrm{d}_A a_{(2)})-(-1)^{|\eta|+|\xi|}((\mathrm{d}_B\eta)\triangleleft a_{(1)})(\xi\triangleleft a_{(2)})-(-1)^{|\xi|}(\eta\triangleleft a_{(1)})((\mathrm{d}_B\xi)\triangleleft a_{(2)})\\
&=(\eta\triangleleft a_{(1)})(\xi\triangleleft\mathrm{d}_A a_{(2)})+(-1)^{|\xi|}(\eta\triangleleft\mathrm{d}_A a_{(1)})(\xi\triangleleft a_{(2)})
\end{aligned}
$$

where we see the coproduct $\Delta_*(\mathrm{d}_A a) = a_{(1)} \otimes \mathrm{d}_A a_{(2)} + \mathrm{d}_A a_{(1)} \otimes a_{(2)}$ of $\Omega(A)$. As a direct consequence, one can find that

$$(\eta\xi)\triangleleft((\mathrm{d}_A a)c) =(\eta\triangleleft(a_{(1)}c_{(1)}))(\xi\triangleleft((\mathrm{d}_A a_{(2)})c_{(2)})) + (-1)^{|\xi|}(\eta\triangleleft((\mathrm{d}_A a_{(1)})c_{(1)}))(\xi\triangleleft(a_{(2)}c_{(2)}))$$

$$(\eta\xi)\triangleleft(a\mathrm{d}_A c) =(\eta\triangleleft(a_{(1)}c_{(1)}))(\xi\triangleleft(a_{(2)}\mathrm{d}_A c_{(2)})) + (-1)^{|\xi|}(\eta\triangleleft(a_{(1)}\mathrm{d}_A c_{(1)}))(\xi\triangleleft(a_{(2)}c_{(2)}))$$

as required for the action of general elements of $\Omega^1(A)$. Since $\Omega(A)$ is the maximal prolongation of $\Omega^1(A)$, then by taking $a\mathrm{d}_A c = 0$, one can check that

$$0 =\mathrm{d}_B\big((\eta\xi)\triangleleft(a\mathrm{d}_A c)\big) = (\mathrm{d}_B(\eta\xi))\triangleleft(a\mathrm{d}_A c) + (-1)^{|\eta|+|\xi|}(\eta\xi)\triangleleft(\mathrm{d}_A a\mathrm{d}_A c)$$

implying $(\eta\xi)\triangleleft(\mathrm{d}_A a\mathrm{d}_A c) = 0$, making $\Omega(B)$ a super right $\Omega(A)$-module algebra.

□

There is an equally good left-handed definition of differentiable action, where the left action $\triangleright : A \otimes B \to B$ extends to $\triangleright : \Omega(A)\underline{\otimes}\Omega(B) \to \Omega(B)$, making $\Omega(B)$ a super left $\Omega(A)$-module algebra.

## 5.2 Differentials by super double cross product

Let $A$ and $H$ be two bialgebras or Hopf algebras with $H$ be a right $A$-module coalgebra by $\triangleleft : H \otimes A \to H$, and $A$ be a left $H$-module coalgebra by $\triangleright : H \otimes A \to A$. We suppose that $\triangleleft$ and $\triangleright$ are compatible as in [19, 23] such that they form a double cross product Hopf algebra $A\bowtie H$.

Now let $\Omega(A)$ and $\Omega(H)$ be strongly bicovariant exterior algebras, and extend $\triangleleft$ and $\triangleright$ to $\triangleleft : \Omega(H)\underline{\otimes}\Omega(A) \to \Omega(H)$ and $\triangleright : \Omega(H)\underline{\otimes}\Omega(A) \to \Omega(A)$ as module coalgebras such that

$$\mathrm{d}_H(\eta\triangleleft\omega) =(\mathrm{d}_H\eta)\triangleleft\omega + (-1)^{|\eta|}\eta\triangleleft\mathrm{d}_A\omega \tag{5.2.1}$$

$$\mathrm{d}_A(\eta\triangleright\omega) =(\mathrm{d}_H\eta)\triangleright\omega + (-1)^{|\eta|}\eta\triangleright\mathrm{d}_A\omega. \tag{5.2.2}$$

If $\triangleleft$ and $\triangleright$ obey the super double cross product conditions:

$$1{\triangleleft}\omega = \epsilon\omega, \quad \eta{\triangleright}1 = \epsilon\eta \tag{5.2.3}$$

$$(\eta\xi){\triangleleft}\omega = (-1)^{|\omega_{(1)}||\xi_{(2)}|}(\eta{\triangleleft}(\xi_{(1)}{\triangleright}\omega_{(1)}))(\xi_{(2)}{\triangleleft}\omega_{(2)}) \tag{5.2.4}$$

$$\eta{\triangleright}(\omega\tau) = (-1)^{|\omega_{(1)}||\eta_{(2)}|}(\eta_{(1)}{\triangleright}\omega_{(1)})((\eta_{(2)}{\triangleleft}\omega_{(2)}){\triangleright}\tau) \tag{5.2.5}$$

$$\begin{aligned}(-1)^{|\omega_{(1)}||\eta_{(2)}|}&\eta_{(1)}{\triangleleft}\omega_{(1)} \otimes \eta_{(2)}{\triangleright}\omega_{(2)} \\ &= (-1)^{|\eta_{(1)}|(|\eta_{(2)}|+|\omega_{(2)}|)+|\omega_{(1)}||\omega_{(2)}|}\eta_{(2)}{\triangleleft}\omega_{(2)} \otimes \eta_{(1)}{\triangleright}\omega_{(1)}\end{aligned} \tag{5.2.6}$$

then we have a double cross product super bialgebra or super Hopf algebra $\Omega(A){\bowtie}\Omega(H)$ with super tensor product coalgebra and the product

$$(\omega \otimes \eta)(\tau \otimes \xi) = (-1)^{|\eta_{(2)}||\tau_{(1)}|}\omega(\eta_{(1)}{\triangleright}\tau_{(1)}) \otimes (\eta_{(2)}{\triangleleft}\tau_{(2)})\xi$$

for all $\eta, \xi \in \Omega(H)$ and $\omega, \tau \in \Omega(A)$. We omit the proof that $\Omega(A){\bowtie}\Omega(H)$ is a super Hopf algebra since this is similar to the usual version [19] with extra signs.

**Theorem 5.2.1.** *Let $A, H$ be bialgebras or Hopf algebras and that form a double cross product $A{\bowtie}H$ and let $\Omega(A)$ and $\Omega(H)$ be strongly bicovariant with $\triangleleft, \triangleright$ obeying the conditions (5.2.1)-(5.2.6). Then $\Omega(A{\bowtie}H) := \Omega(A){\bowtie}\Omega(H)$ is a strongly bicovariant exterior algebra on $A{\bowtie}H$ with differential*

$$\mathrm{d}(\omega \otimes \eta) = \mathrm{d}_A\omega \otimes \eta + (-1)^{|\omega|}\omega \otimes \mathrm{d}_H\eta$$

*Proof.* First note that $\Omega^1(A{\bowtie}H) = \mathrm{span}\{(a\otimes h)\mathrm{d}(b\otimes g)\} = \mathrm{span}\{a\mathrm{d}_Ab\otimes f + c\otimes h\mathrm{d}_Hg\} = \Omega^1(A){\bowtie}H \oplus A{\bowtie}\Omega^1(H)$ since

$$(c \otimes h)\mathrm{d}(1 \otimes g) = c \otimes h\mathrm{d}_Hg, \quad (a \otimes 1)\mathrm{d}(b \otimes f) - (ab \otimes 1)\mathrm{d}(1 \otimes f) = a\mathrm{d}_Ab \otimes f$$

for all $a, b, c \in A$ and $f, g, h \in H$.

Since $\mathrm{d}(\omega \otimes 1) = \mathrm{d}_A\omega$ and $\mathrm{d}(1 \otimes \eta) = \mathrm{d}_H\eta$ for all $\eta \in \Omega(H)$ and $\omega \in \Omega(A)$, we show that the graded Leibniz rule holds as

$$\begin{aligned}
\mathrm{d}(\eta\omega) \equiv & \mathrm{d}((1\otimes\eta)(\omega\otimes 1)) \\
= & (-1)^{|\eta_{(2)}||\omega_{(1)}|}\big((\mathrm{d}_H\eta_{(1)}){\triangleright}\omega_{(1)}\otimes\eta_{(2)}{\triangleleft}\omega_{(2)} + (-1)^{|\eta_{(1)}|+|\omega_{(1)}|}\eta_{(1)}{\triangleright}\omega_{(1)}\otimes(\mathrm{d}_H\eta_{(2)}){\triangleleft}\omega_{(2)}\big) \\
& + (-1)^{|\eta_{(2)}||\omega_{(1)}|+|\eta_{(1)}|}\big(\eta_{(1)}{\triangleright}\mathrm{d}_A\omega_{(1)}\otimes\eta_{(2)}{\triangleleft}\omega_{(2)} + (-1)^{|\eta_{(2)}|+|\omega_{(1)}|}\eta_{(1)}{\triangleright}\omega_{(1)}\otimes\eta_{(2)}{\triangleleft}\mathrm{d}_A\omega_{(2)}\big) \\
= & (-1)^{|(\mathrm{d}_H\eta)_{(2)}||\omega_{(1)}|}(\mathrm{d}_H\eta)_{(1)}{\triangleright}\otimes\omega_{(1)}\otimes(\mathrm{d}_H\eta)_{(2)}{\triangleleft}\omega_{(2)} \\
& + (-1)^{|\eta|+|\eta_{(1)}||(\mathrm{d}_A\omega)_{(1)}|}\eta_{(1)}{\triangleright}(\mathrm{d}_A\omega)_{(1)}\otimes\eta_{(2)}{\triangleleft}(\mathrm{d}_A\omega)_{(2)} \\
= & (1\otimes\mathrm{d}_H\eta)(\omega\otimes 1) + (-1)^{|\eta|}(1\otimes\eta)(\mathrm{d}_A\omega\otimes 1) \equiv (\mathrm{d}\eta)\omega + (-1)^{|\eta|}\eta\mathrm{d}\omega.
\end{aligned}$$

Clearly $\mathrm{d}^2 = 0$ and thus $\Omega(A){\bowtie}\Omega(H)$ is a DGA. Finally since $\Delta_*$ is a super tensor coproduct as in Lemma 2.5.7, then d is a super-coderivation .

$\square$

**Remark 5.2.2.** If $A$ is finite-dimensional, it is explained in [38] that $A{\bowtie}H$ acts on $A^*$ as module algebra by

$$(\phi{\triangleleft}h)(a) = \phi(h{\triangleright}a), \quad \phi{\triangleleft}a = \langle\phi_{(1)}, a\rangle\phi_{(2)},$$

for all $\phi\in A^*$, $a\in A$, and $h\in H$. Similarly for a left action on $H^*$. However, for differentiablity, we would need $\Omega(A^*)$ or $\Omega(H^*)$ to be specified.

### 5.2.1 Exterior algebra on generalised quantum double $D(A,H)$

Let $A, H$ be dually paired Hopf algebras with duality pairing $\langle\ ,\ \rangle : H\otimes A\to k$. Then, $A$ acts on $H$ and $H$ acts on $A$ by the following actions

$$h{\triangleleft}a = h_{(2)}\langle Sh_{(1)}, a_{(1)}\rangle\langle h_{(3)}, a_{(2)}\rangle, \quad h{\triangleright}a = a_{(2)}\langle Sh_{(1)}, a_{(1)}\rangle\langle h_{(2)}, a_{(3)}\rangle,$$

and one has a generalised quantum double $D(A,H) = A^{\mathrm{op}}{\bowtie}H$ [19]. In this case, the product becomes $(a\otimes b)(c\otimes d) = \langle Sb_{(1)}, c_{(1)}\rangle c_{(2)}a\otimes b_{(2)}d\langle b_{(3)}, c_{(3)}\rangle$. Note that if $A = H^*$ is finite-dimensional, then we recover Drinfeld's double [11].

Note that if $\Omega(A)$ is a DGA, then $\Omega(A)^{\mathrm{op}}$ remains a DGA since

$$
\begin{aligned}
\mathrm{d}_A(\omega \cdot_{\mathrm{op}} \tau) =&(-1)^{|\tau||\omega|}\mathrm{d}_A(\tau\omega) = (-1)^{|\tau||\omega|}((\mathrm{d}_A\tau)\omega + (-1)^{|\tau|}\tau\mathrm{d}_A\omega)\\
=&(-1)^{|\tau|(|\omega|+1)}\tau\mathrm{d}_A\omega + (-1)^{(|\tau|+1)|\omega|+|\omega|}(\mathrm{d}_A\tau)\omega\\
=&\mathrm{d}_A\omega \cdot_{\mathrm{op}} \tau + (-1)^{|\omega|}\omega \cdot_{\mathrm{op}} \mathrm{d}_A\tau,
\end{aligned}
$$

for all $\omega, \tau \in \Omega(A)$. Therefore we can define $\Omega(A^{\mathrm{op}}) := \Omega(A)^{\mathrm{op}}$ with differential $\mathrm{d}_A$. Now let $\Omega(H), \Omega(A)$ be a strongly bicovariant exterior algebras and suppose that the above pairing can be extended to a super Hopf algebra pairing $\langle\ ,\ \rangle : \Omega(H) \otimes \Omega(A) \to k$ by 0 for degree $\geq 1$. So we have

$$
\eta \triangleleft \omega = \begin{cases} \eta_{(2)}\langle S\eta_{(1)}, \omega_{(1)}\rangle\langle \eta_{(3)}, \omega_{(2)}\rangle & \text{if } \omega \in A\\ 0 & \text{otherwise,}\end{cases}
$$

$$
\eta \triangleright \omega = \begin{cases} \omega_{(2)}\langle S\eta_{(1)}, \omega_{(1)}\rangle\langle \eta_{(2)}, \omega_{(3)}\rangle & \text{if } \eta \in H\\ 0 & \text{otherwise,}\end{cases}
$$

where only the part of $\Delta_*^2\eta$ in $H \otimes \Omega(H) \otimes H$ contributes in the first action, and only the part of $\Delta_*^2\omega$ in $A \otimes \Omega(A) \otimes A$ contributes in the second action.

One can check that the above actions obey conditions (5.2.1) – (5.2.6) for super double cross product. Therefore, there is a super double cross product $\Omega(A)^{\mathrm{op}}\bowtie\Omega(H)$ with product

$$
(\omega \otimes \eta)(\tau \otimes \xi) = (-1)^{(|\eta|+|\omega|)|\tau|}\langle S\eta_{(1)}, \tau_{(1)}\rangle \tau_{(2)}\omega \otimes \eta_{(2)}\xi\langle \eta_{(3)}, \tau_{(3)}\rangle
$$

for all $\omega, \tau \in \Omega(A)$, $\eta, \xi \in \Omega(H)$ and super tensor product coalgebra. Note that in the above product, $\tau_{(2)}$ and $\omega$ are crossed with $\eta_{(2)}$ and so we should have generated a factor $(-1)^{(|\eta_{(2)}|+|\omega|)|\tau_{(2)}|}$ but $\langle\ ,\ \rangle$ is 0 on degree $\geq 1$, and the part of $\Delta_*^2\eta$ in $H \otimes \Omega(H) \otimes H$ and the part of $\Delta_*^2\tau$ in $A \otimes \Omega(A) \otimes A$ contribute, so $|\eta_{(2)}| = |\eta|$ and $|\tau_{(2)}| = |\tau|$.

**Proposition 5.2.3.** *Let $A, H$ be dually paired Hopf algebras forming $D(A, H) = A^{\mathrm{op}}\bowtie H$.*

*Let $\Omega(A)$ and $\Omega(H)$ be dually paired strongly bicovariant exterior algebras as above. Then $\Omega(D(A,H)) := \Omega(A)^{\mathrm{op}}\bowtie\Omega(H)$ is a strongly bicovariant exterior algebra on the generalised quantum double $D(A,H)$ with differential*

$$\mathrm{d}(\omega\otimes\eta) = \mathrm{d}_A\omega\otimes\eta + (-1)^{|\omega|}\omega\otimes\mathrm{d}_H\eta,$$

*for all $\eta\in\Omega(H)$ and $\omega\in\Omega(A)$. Moreover, the action of $D(A,H)$ on $H$ defined by $g{\triangleleft}(a\otimes h) = \langle g_{(2)},a\rangle(Sh_{(1)})g_{(1)}h_{(2)}$ extends to a differentiable action of $\Omega(D(A,H))$ on $\Omega(H)$ by*

$$\xi{\triangleleft}(\omega\otimes\eta) = \begin{cases}(-1)^{|\eta_{(1)}||\xi|}\langle\xi_{(2)},\omega\rangle(S\eta_{(1)})\xi_{(1)}\eta_{(2)} & \text{if } \omega\in A\\ 0 & \text{otherwise},\end{cases}$$

*where only the part of $\Delta_*\xi$ in $\Omega(H)\otimes H$ contributes. Explicitly*

$$\xi{\triangleleft}(a\otimes\eta) = (-1)^{|\eta_{(1)}||\xi|}\langle\xi^{\overline{(1)}},a\rangle(S\eta_{(1)})\xi^{\overline{(0)}}\eta_{(2)},$$

*where $\Delta_R\xi = \xi^{\overline{(0)}}\otimes\xi^{\overline{(1)}}$ is the right coaction of $H$ on $\Omega(H)$. Similarly with left-right reversal for a differentiable left action of $D(A,H)$ on $A$.*

*Proof.* The first part of the proposition follows from Theorem 5.2.1. One can check the stated action makes $\Omega(H)$ a super right $\Omega(D(A,H))$-module algebra. Moreover, $\Delta_*\xi = \Delta_R\xi +$ terms of higher degree on the second factor, giving the explicit formula stated. It also respects d since

$$\begin{aligned}\mathrm{d}_H(\xi{\triangleleft}(a\otimes\eta)) =&(-1)^{|\eta_{(1)}||\xi|}\langle\xi^{\overline{(1)}},a\rangle\mathrm{d}_H((S\eta_{(1)})\xi^{\overline{(0)}}\eta_{(2)})\\ =&(-1)^{|\eta_{(1)}||\xi|}\langle\xi^{\overline{(1)}},a\rangle\Big((\mathrm{d}_H S\eta_{(1)})\xi^{\overline{(0)}}\eta_{(2)} + (-1)^{|\eta_{(1)}|}(S\eta_{(1)})(\mathrm{d}_H\xi^{\overline{(0)}})\eta_{(2)}\\ &+ (S\eta_{(1)})\xi^{\overline{(0)}}\mathrm{d}_H\eta_{(2)}\Big)\\ =&(\mathrm{d}_H\xi){\triangleleft}a\otimes\eta + (-1)^{|\xi|}\xi{\triangleleft}\mathrm{d}(a\otimes\eta),\end{aligned}$$

where in the last equation, we have $\xi{\triangleleft}\mathrm{d}(a\otimes\eta)=\xi{\triangleleft}(a\otimes \mathrm{d}_H\eta)$ since $\xi{\triangleleft}(\mathrm{d}_A a\otimes\eta)=0$. The formulae with $A, H$ and left-right reversal and its proof are similar. □

**Example 5.2.4.** Let $H=U_q(b_+)$ be a self-dual Hopf algebra generated by $x, t$ with relations, comultiplication, and duality pairing

$$tx=q^2xt,\quad \Delta t=t\otimes t,\quad \Delta x=1\otimes x+x\otimes t$$

$$\langle t,s\rangle=q^{-2},\quad \langle x,s\rangle=\langle t,y\rangle=0,\quad \langle x,y\rangle=\frac{1}{1-q^2}$$

where $y, s$ are another copies of $x, t$, regarded as generators of $A^{\mathrm{op}}=U_q(b_+)^{\mathrm{op}}=U_{q^{-1}}(b_+)$ and $q^2\neq 1$. Let $\Omega(U_q(b_+))$ be strongly bicovariant (it will be constructed later in Proposition 6.4.1) with the following bimodule relations and comultiplication

$$(\mathrm{d}t)t=q^2t\mathrm{d}t,\quad (\mathrm{d}x)x=q^2x\mathrm{d}x,\quad (\mathrm{d}x)t=t\mathrm{d}x,\quad (\mathrm{d}t)x=q^2x\mathrm{d}t+(q^2-1)t\mathrm{d}x$$

$$(\mathrm{d}t)^2=(\mathrm{d}x)^2=0,\quad \mathrm{d}t\mathrm{d}x=-\mathrm{d}x\mathrm{d}t,\quad \Delta_*\mathrm{d}t=\mathrm{d}t\otimes t+t\otimes\mathrm{d}t,\quad \Delta_*\mathrm{d}x=1\otimes\mathrm{d}x+\mathrm{d}x\otimes t+x\otimes\mathrm{d}t.$$

Then $\Omega(D(U_q(b_+)))$ contains $\Omega(U_q(b_+))$ and $\Omega(U_{q^{-1}}(b_+))$ as sub-strongly bicovariant exterior algebras, and the following cross-relations

$$ts=st,\quad ty=q^{-2}yt,\quad xs=q^{-2}sx,\quad xy=q^{-2}yx+\frac{1-st}{1-q^2}$$

$$(\mathrm{d}t)s=s\mathrm{d}t,\quad (\mathrm{d}s)t=t\mathrm{d}s,\quad (\mathrm{d}x)s=q^{-2}s\mathrm{d}s,\quad (\mathrm{d}s)x=q^2x\mathrm{d}s,\quad (\mathrm{d}t)y=q^{-2}y\mathrm{d}t,$$

$$(\mathrm{d}y)t=q^2t\mathrm{d}y,\quad (\mathrm{d}x)y=q^{-2}y\mathrm{d}x-\frac{s\mathrm{d}t}{1-q^2},\quad (\mathrm{d}y)x=q^2x\mathrm{d}y+\frac{t\mathrm{d}s}{q^{-2}-1},$$

$$\mathrm{d}t\mathrm{d}s=-\mathrm{d}s\mathrm{d}t,\quad \mathrm{d}t\mathrm{d}y=-q^{-2}\mathrm{d}y\mathrm{d}t,\quad \mathrm{d}x\mathrm{d}s=-q^{-2}\mathrm{d}s\mathrm{d}x,\quad \mathrm{d}x\mathrm{d}y=-q^{-2}\mathrm{d}y\mathrm{d}x-\frac{\mathrm{d}s\mathrm{d}t}{1-q^2}.$$

Moreover, $\Omega(D_q(U(b_+)))$ acts differentiably on $U_q(b_+)$.

*Proof.* One can check that $\langle\ ,\ \rangle$ extends by 0 on degree $\geq 1$. Then the stated crossed relations can be found by direct calculation and one can check that the graded Leibniz rule holds, and d is a super-coderivation. By Proposition 5.2.3, $\Omega(U_{q^{-1}}(b_+)){\bowtie\!\!\!\triangleleft}\Omega(U_q(b_+))$ acts differentiably by

$$t\triangleleft t = t,\quad t\triangleleft x = (1-q^{-2})tx,\quad x\triangleleft t = q^{-2}x,\quad x\triangleleft x = (1-q^{-2})x^2,$$

$$(\mathrm{d}t)\triangleleft t = q^2\mathrm{d}t,\quad (\mathrm{d}t)\triangleleft x = (q^2-1)t\mathrm{d}x,\quad (\mathrm{d}x)\triangleleft t = \mathrm{d}x,\quad (\mathrm{d}x)\triangleleft x = (q^2-1)x\mathrm{d}x,$$

$$t\triangleleft \mathrm{d}t = (1-q^2)\mathrm{d}t,\quad t\triangleleft \mathrm{d}x = (q^2-1)x\mathrm{d}t,\quad x\triangleleft \mathrm{d}t = (q^{-2}-1)\mathrm{d}x,\quad x\triangleleft \mathrm{d}x = (1-q^{-2})x\mathrm{d}x,$$

$$(\mathrm{d}t)\triangleleft \mathrm{d}t = (\mathrm{d}x)\triangleleft \mathrm{d}t = (\mathrm{d}x)\triangleleft \mathrm{d}x = 0,\quad (\mathrm{d}t)\triangleleft \mathrm{d}x = (1-q^2)\mathrm{d}t\mathrm{d}x,$$

$$t\triangleleft s = q^{-2}t,\quad t\triangleleft y = 0,\quad x\triangleleft s = x,\quad x\triangleleft y = \frac{t}{1-q^2},$$

$$(\mathrm{d}t)\triangleleft s = q^{-2}\mathrm{d}t,\quad (\mathrm{d}t)\triangleleft y = 0,\quad (\mathrm{d}x)\triangleleft s = \mathrm{d}x,\quad (\mathrm{d}x)\triangleleft y = \frac{\mathrm{d}t}{1-q^2},$$

$$t\triangleleft \mathrm{d}s = t\triangleleft \mathrm{d}y = x\triangleleft \mathrm{d}s = x\triangleleft \mathrm{d}y = 0.$$

□

**Remark 5.2.5.** It is known [11] that $D(U_q(b_+))/(st^{-1}-1) \cong U_q(sl_2)$ by

$$x \mapsto x_+K,\quad y\mapsto x_-K,\quad t\mapsto K^2,$$

where here $U_q(sl_2)$ is generated by $K, x_\pm$ with the following relation

$$KK^{-1} = K^{-1}K,\quad Kx_\pm = q^{\pm 1}x_\pm K,\quad [x_+,x_-] = \frac{K^2-K^{-2}}{q-q^{-1}}.$$

However, $\Omega(D(U_q(b_+)))$ in Example 5.2.4 above does not descend to $\Omega(U_q(sl_2))$ since $\mathrm{d}(st^{-1}) = 0$ gives $\mathrm{d}s = q^{-2}\mathrm{d}t$.

**Example 5.2.6.** Let $U(su_2)$ be the enveloping algebra of $su_2$ with generators $x_a$, for $a=1,2,3$ with primitive coproducts and relations $[x_a,x_b] = 2\lambda\epsilon_{abc}x_c$ where $\lambda\in \imath\mathbb{R}$ is a purely imaginary number and $\epsilon_{abc}$ is the totally antisymmetric tensor. Let $\Omega(U(su_2))$

be a 4D strongly bicovariant exterior algebra with the following bimodule relations [4]

$$[\mathrm{d}x_a, x_b] = \lambda\epsilon_{abc}\mathrm{d}x_c - \lambda^2\delta^a{}_b\theta, \quad [\theta, x_a] = \mathrm{d}x_a$$

$$\mathrm{d}\theta = 0, \quad \{\mathrm{d}x_a, \mathrm{d}x_b\} = 0, \quad \{\theta, \mathrm{d}x_a\} = 0,$$

and primitives coproducts on $\mathrm{d}x_a$ and $\theta$.

Let $\mathbb{C}[SU_2]$ be the commutative Hopf algebra generated as usual by $\mathbf{t} = (t^i{}_j)$, with determinant $t^1{}_1t^2{}_2 - t^1{}_2t^2{}_1 = 1$ and $\Delta t^i{}_j = t^i{}_k \otimes t^k{}_j$. Also let $\Omega(\mathbb{C}[SU_2])$ be the classical 3D strongly bicovariant exterior algebra with

$$[\mathrm{d}t^i{}_j, t^k{}_l] = 0, \quad \{\mathrm{d}t^i{}_j, \mathrm{d}t^k{}_l\} = 0, \quad \mathrm{d}t^2{}_2 = (t^2{}_2)(t^1{}_2\mathrm{d}t^2{}_1 + t^2{}_1\mathrm{d}t^1{}_2 - t^2{}_2\mathrm{d}t^1{}_1),$$

$$\Delta_*\mathrm{d}t^i{}_j = \mathrm{d}t^i{}_k \otimes t^k{}_j + t^i{}_k \otimes \mathrm{d}t^k{}_j,$$

and $S\mathbf{t} = \mathbf{t}^{-1}$ as usual. Then $\Omega(\mathbb{C}[SU_2]){>\!\!\triangleleft}\Omega(U(su_2))$ is strongly bicovariant and contains $\Omega(\mathbb{C}[SU_2])$ and $\Omega(U(su_2))$ as sub-strongly bicovariant exterior algebra, with the following cross bimodule relations

$$[x_a, t^i{}_j] = -\imath\lambda(t^i{}_k(\sigma_a)^k{}_j - (\sigma_a)^i{}_kt^k{}_j), \quad [x_a, dt^i{}_j] = -\imath\lambda(\mathrm{d}t^i{}_k(\sigma_a)^k{}_j - (\sigma_a)^i{}_k\mathrm{d}t^k{}_j)$$

$$[\mathrm{d}x_a, t^i{}_j] = 0, \quad [\theta, t^i{}_j] = 0, \quad \{\mathrm{d}x_a, \mathrm{d}t^i{}_j\} = 0, \quad \{\theta, \mathrm{d}t^i{}_j\} = 0,$$

where $(\sigma_a)^i{}_j$ is the $(i,j)$-th entry of the standard Pauli matrix $\sigma_a$ for $a = 1, 2, 3$. Moreover, $\Omega(\mathbb{C}[SU_2]){>\!\!\triangleleft}\Omega(U(su_2))$ acts differentiably on $U(su_2)$, and it forms a $*$-differential calculus with the usual $*$-structure of $U(su_2)$ and $\mathbb{C}[SU_2]$.

*Proof.* First note that $x_a{\triangleleft}t = x_a$ for all $t \in \mathbb{C}[SU_2]$, so $\mathbb{C}[SU_2]{\bowtie}U(su_2) = \mathbb{C}[SU_2]{>\!\!\triangleleft}U(su_2)$. The duality pairing between $\mathbb{C}[SU_2]$ and $U(su_2)$ is given by $\langle t^i{}_j, x_a\rangle = -\imath\lambda(\sigma_a)^i{}_j$, and this gives the stated cross relation on degree 0 as found previously in [4]. One can check that $\langle\ ,\ \rangle$ extends to the pairing between $\Omega(\mathbb{C}[SU_2])$ and $\Omega(U(su_2))$ by 0 for degree $\geq 1$, giving the rest of stated crossed relations on $\Omega(\mathbb{C}[SU_2])\rtimes\!\!\!\lhd\Omega(U(su_2))$. One can also check that the graded Leibniz rule holds and d is a super-coderivation. Note that $\Omega(\mathbb{C}[SU_2])$ and $\Omega(U(su_2))$ are $*$-calculi (in the usual sense that $*$ commutes with d and is a graded antilinear order-reversing involution), with the usual $*$-structure given by

$$x_a^* = x_a, \quad \theta^* = -\theta, \quad (\mathrm{d}x_a)^* = \mathrm{d}x_a, \quad (t^i{}_j)^* = St^j_i, \quad (\mathrm{d}t^i{}_j)^* = S\mathrm{d}t^j{}_i$$

We check that these results in a $*$-calculus, e.g.

$$\begin{aligned}[x_a, \mathrm{d}t^i{}_j]^* =&[(\mathrm{d}t^i{}_j)^*, x_a^*] = [S\mathrm{d}t^i{}_j, x_a] = -\imath\lambda((\sigma_a)^j{}_k S\mathrm{d}t^k{}_i - (S\mathrm{d}t^j{}_k)(\sigma_a)^k{}_i)\\ =&\big(-\imath\lambda(\mathrm{d}t^i{}_k(\sigma_a)^k{}_j - (\sigma_a)^i{}_k\mathrm{d}t^k{}_j)\big)^*.\end{aligned}$$

By Proposition 5.2.3, the action of $\Omega(\mathbb{C}[SU_2])\rtimes\!\!\!\lhd\Omega(U(su_2))$ on $\Omega(U(su_2))$ as a super module algebra is given by

$$x_a{\triangleleft}x_b = [x_a, x_b], \quad (\mathrm{d}x_a){\triangleleft}x_b = [\mathrm{d}x_a, x_b], \quad x_a{\triangleleft}\mathrm{d}x_b = [x_a, \mathrm{d}x_b], \quad (\mathrm{d}x_a){\triangleleft}\mathrm{d}x_b = \{\mathrm{d}x_a, \mathrm{d}x_b\},$$

$$x_a{\triangleleft}t^i{}_j = x_a\delta^i{}_j - \imath\lambda(\sigma_a)^i{}_j, \quad (\mathrm{d}x_a){\triangleleft}t^i{}_j = \mathrm{d}x_a\delta^i{}_j, \quad x_a{\triangleleft}\mathrm{d}t^i{}_j = 0$$

$$\theta{\triangleleft}t^i{}_j = \theta\delta^i{}_j, \quad (\mathrm{d}x_a){\triangleleft}\mathrm{d}t^i{}_j = 0, \quad \theta{\triangleleft}\mathrm{d}t^i{}_j = 0.$$

□

**Remark 5.2.7.** One can replace $A^{\mathrm{op}}$ by $A$ in the above construction and regard the Hopf algebra pairing $\langle\ ,\ \rangle$ as a skew pairing $\sigma : H\otimes A \to k$, that is an involutive-invertible map satisfying

$$\sigma(hg, a) = \sigma(h, a_{(1)})\sigma(g, a_{(2)}), \quad \sigma(h, ab) = \sigma(h_{(2)}, a)\sigma(h_{(1)}, b).$$

Here $\langle S(\ ),\ \rangle$ provides $\sigma^{-1}$ if we start with a Hopf algebra pairing. The above is then equivalent to a generalised quantum double $A\bowtie_\sigma H$. By extending to $\sigma : \Omega(H)\underline{\otimes}\Omega(A) \to k$ by 0 for degree $\geq 1$, we have a super double cross product $\Omega(A)\bowtie_\sigma\Omega(H)$, and by

Theorem 5.2.1 we have $\Omega(A{\bowtie}_\sigma H) := \Omega(A){\bowtie}_\sigma\Omega(H)$. In this approach, we can work with strongly bicovariant exterior algebra $\Omega(A)$ rather than with $\Omega(A)^{\mathrm{op}}$.

### 5.2.2 Exterior algebra on $A{\bowtie}_{\mathcal{R}}A$

Let $(A,\mathcal{R})$ be a coquasitriangular Hopf algebra. In this case, we can view $A$ as skew-paired with itself by $\sigma=\mathcal{R}$ in Remark 5.2.7 and have the double coquasitriangular $A{\bowtie}_{\mathcal{R}}A$ with $A$ left and right acts on itself by

$$b{\triangleleft}a = b_{(2)}\mathcal{R}(Sb_{(1)},a_{(1)})\mathcal{R}(b_{(3)},a_{(2)}),\quad b{\triangleright}a = a_{(2)}\mathcal{R}(Sb_{(1)},a_{(1)})\mathcal{R}(b_{(2)},a_{(3)}).$$

Here the product is $(a\otimes b)(c\otimes d)=\mathcal{R}(Sb_{(1)},c_{(1)})ac_{(2)}\otimes b_{(2)}d\mathcal{R}(b_{(3)},c_{(3)})$ for all $a,b,c,d\in A$, see [19]. For example, if $A=\mathbb{C}_q[SU_2]$ then this can be viewed as $\mathbb{C}_q[SO_{3,1}]$, if we work over $\mathbb{C}$ with the relevant $*$-structures.

Let $\Omega(A)$ be a strongly bicovariant exterior algebra, viewed as a super-coquasitriangular Hopf algebra with $\mathcal{R}$ extended by 0 on degree $\geq 1$. The above actions extend to actions of $\Omega(A)$ on itself by

$$\eta{\triangleleft}\omega=\begin{cases}\eta_{(2)}\mathcal{R}(S\eta_{(1)},\omega_{(1)})\mathcal{R}(\eta_{(3)},\omega_{(2)}) & \text{if } \omega\in A\\ 0 & \text{otherwise,}\end{cases}$$
$$\eta{\triangleright}\omega=\begin{cases}\omega_{(2)}\mathcal{R}(S\eta_{(1)},\omega_{(1)})\mathcal{R}(\eta_{(2)},\omega_{(3)}) & \text{if } \eta\in A\\ 0 & \text{otherwise,}\end{cases}$$

where only the part of $\Delta_*^2\eta\in A\otimes\Omega(A)\otimes A$ contributes in the first action, and only the part of $\Delta_*^2\omega\in A\otimes\Omega(A)\otimes A$ contributes in the second action. One can check that these actions obey (5.2.1) – (5.2.6) and thus we have a super double version $\Omega(A){\bowtie}_{\mathcal{R}}\Omega(A)$ with product coproduct

$$(\omega\otimes\eta)(\tau\otimes\xi)=(-1)^{|\eta||\tau|}\mathcal{R}(S\eta_{(1)},\tau_{(1)})\omega\tau_{(2)}\otimes\eta_{(2)}\xi\mathcal{R}(\eta_{(3)},\tau_{(3)})$$

$$\Delta_*(\omega\otimes\eta) = (-1)^{|\eta_{(1)}||\omega_{(2)}|}\omega_{(1)}\otimes\eta_{(1)}\otimes\omega_{(2)}\otimes\eta_{(2)}.$$

for all $\omega,\eta,\tau,\xi\in\Omega(A)$. Note that in the above product, the crossing between $\eta_{(2)}$ and $\tau_{(2)}$ so we should have generated a factor $(-1)^{|\eta_{(2)}||\tau_{(2)}|}$ but $\mathcal{R}$ is 0 on degree $\geq 1$, so only the parts of $\Delta_*^2\eta$ and $\Delta_*^2\tau$ in $A\otimes\Omega(A)\otimes A$ contributes, resulting in $|\eta_{(2)}|=|\eta|$ and $|\tau_{(2)}|=|\tau|$.

**Corollary 5.2.8.** *Let $\Omega(A)$ be a strongly bicovariant exterior algebra on a coquasitriangular Hopf algebra $A$. Then $\Omega(A{\bowtie_{\mathcal{R}}}A) := \Omega(A){\bowtie_{\mathcal{R}}}\Omega(A)$ is a strongly bicovariant exterior algebra on $A{\bowtie_{\mathcal{R}}}A$ with differential*

$$\mathrm{d}(\omega\otimes\eta) = \mathrm{d}_A\omega\otimes\eta + (-1)^{|\omega|}\omega\otimes\mathrm{d}_A\eta,$$

*and $A{\bowtie_{\mathcal{R}}}A$ acts on $A$ given by $a{\triangleleft}(b\otimes c) = \mathcal{R}(a_{(2)},b)(Sc_{(1)})a_{(1)}c_{(2)}$ for all $a,b,c\in A$ extends to an action of $\Omega(A){\bowtie_{\mathcal{R}}}\Omega(A)$ differentiably with*

$$\tau{\triangleleft}(\omega\otimes\eta) = \begin{cases}(-1)^{|\eta_{(1)}||\tau|}\mathcal{R}(\tau_{(2)},\omega)(S\eta_{(1)})\tau_{(1)}\eta_{(2)} & \text{if } \omega\in A\\ 0 & \text{otherwise,}\end{cases}$$

*where only the part of $\Delta_*\tau\in A\otimes\Omega(A)$ contributes. Moreover, there is a coaction $\Delta_R:\underline{A}\to\underline{A}\otimes A{\bowtie_{\mathcal{R}}}A$, where $\underline{A}$ is a transmutation of $A$, given by $\Delta_R a = a_{(2)}\otimes Sa_{(1)}\otimes a_{(3)}$ which is differentiable with $\Delta_{R*}\omega = (-1)^{|\omega_{(1)}||\omega_{(2)}|}\omega_{(2)}\otimes S\omega_{(1)}\otimes\omega_{(3)}$ for a certain $\Omega(\underline{A})$ (see Section 6.1 and Remark 6.2.4)*

*Proof.* The first part of the statement follows from Proposition 5.2.3, where $\langle\ ,\ \rangle$ is replaced by $\mathcal{R}$, and $\langle S(\ ),\ \rangle$ is replaced by $\mathcal{R}^{-1}$ as in Remark 5.2.7. For the new part, since $\Omega(A)$ is strongly bicvoariant, we defer the proof that $\Delta_{R*}:\Omega(\underline{A})\to\Omega(\underline{A})\underline{\otimes}\Omega(A){\bowtie_{\mathcal{R}}}\Omega(A)$ above is globally defined and makes $\Omega(\underline{A})$ a super $\Omega(A){\bowtie_{\mathcal{R}}}\Omega(A)$-comodule algebra. It also respects d since

$$\Delta_{R*}(\mathrm{d}_A\omega) =(-1)^{|(\mathrm{d}_A\omega)_{(1)}||(\mathrm{d}_A\omega)_{(2)}|}(\mathrm{d}_A\omega)_{(2)}\otimes S(\mathrm{d}_A\omega)_{(1)}\otimes(\mathrm{d}_A\omega)_{(3)}$$

$$
\begin{aligned}
=&(-1)^{|\mathrm{d}_A\omega_{(1)}||\omega_{(2)}|}\omega_{(2)}\otimes S\mathrm{d}_A\omega_{(1)}\otimes\omega_{(3)}+(-1)^{|\omega_{(1)}||\mathrm{d}_A\omega_{(2)}|+|\omega_{(1)}|}\mathrm{d}_A\omega_{(2)}\otimes S\omega_{(1)}\otimes\omega_{(3)}\\
&+(-1)^{|\omega_{(1)}||\omega_{(2)}|+|\omega_{(1)}|+|\omega_{(2)}|}\omega_{(2)}\otimes S\omega_{(1)}\otimes\mathrm{d}_A\omega_{(3)}\\
=&(-1)^{|\omega_{(1)}||\omega_{(2)}|}(\mathrm{d}_A\omega_{(2)}\otimes S\omega_{(1)}\otimes\omega_{(3)}+(-1)^{|\omega_{(2)}|}\omega_{(2)}\otimes\mathrm{d}(S\omega_{(1)}\otimes\omega_{(3)}))\\
=&(\mathrm{d}_A\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_{R*}\omega.
\end{aligned}
$$

□

**Example 5.2.9.** Let $R\in M_n(R)\otimes M_n(R)$ be a $q$-Hecke $R$-matrix, and let $A(R)$ be an FRT bialgebra [12] generated by $\mathbf{t}=(t^i{}_j)$ with relation $R\mathbf{t}_1\mathbf{t}_2=\mathbf{t}_2\mathbf{t}_1R$, and let $\Omega(A(R))$ be strongly bicovariant with relations $(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2=R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R$ and $\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2=-R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R$. Assume that there is a grouplike and central element $D$ in $A(R)$, and let $A=A(R)[D^{-1}]$ be a Hopf algebra localisation of $A(R)$ such that $\Omega(A(R))$ extends to $\Omega(A)$ with $\mathrm{d}D^{-1}=-D^{-1}(\mathrm{d}D)D^{-1}$ (the detail on FRT bialgebra and its strongly bicovariant exterior algebra will be discussed later in Section 6.3). Then $\Omega(A{\bowtie}_{\mathcal{R}}A):=\Omega(A){\bowtie}_{\mathcal{R}}\Omega(A)$ has the following cross relation and comultiplication

$$R\mathbf{t}_1\mathbf{s}_2=\mathbf{s}_2\mathbf{t}_1R,\quad(\mathrm{d}\mathbf{s}_2)\mathbf{t}_1R=R\mathbf{t}_1\mathrm{d}\mathbf{s}_2,\quad R(\mathrm{d}\mathbf{t}_1)\mathbf{s}_2=\mathbf{s}_2\mathrm{d}\mathbf{t}_1R,\quad R\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{s}_2=-\mathrm{d}\mathbf{s}_2\mathrm{d}\mathbf{t}_1R$$

$$\Delta\mathbf{t}=\mathbf{t}\otimes\mathbf{t},\quad\Delta\mathbf{s}=\mathbf{s}\otimes\mathbf{s},\quad\Delta_*\mathrm{d}\mathbf{t}=\mathrm{d}\mathbf{t}\otimes\mathbf{t}+\mathbf{t}\otimes\mathrm{d}\mathbf{t},\quad\Delta_*\mathrm{d}\mathbf{s}=\mathrm{d}\mathbf{s}\otimes\mathbf{s}+\mathbf{s}\otimes\mathrm{d}\mathbf{s},$$

where $\mathbf{s}$ is the copy of $\mathbf{t}$. Furthermore $\Omega(A{\bowtie}_{\mathcal{R}}A)$ acts on $A$ and coacts on $\underline{A}$ differentiably.

*Proof.* One can find the stated relations since $A$ is a coquasitriangular with $\mathcal{R}(\mathbf{t}_1,\mathbf{t}_2)=R$ and $\mathcal{R}(S\mathbf{t}_1,\mathbf{t}_2)=R^{-1}$, and $\mathcal{R}$ extends as super coquasitriangular structure on $\Omega(A(R))$ by zero in degree $\geq 1$. One can check that Leibniz rule hold and d is a super-coderivation. By Corollary 5.2.8, the action of $\Omega(A){\bowtie}_{\mathcal{R}}\Omega(A)$ on $\Omega(A)$ as a super module algebra is given by

$$\mathbf{t}_1{\triangleleft}\mathbf{t}_2=(S\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2,\quad(\mathrm{d}\mathbf{t}_1){\triangleleft}\mathbf{t}_2=(S\mathbf{t}_2)(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2,\quad\mathbf{t}_1{\triangleleft}\mathrm{d}\mathbf{t}_2=(S\mathrm{d}\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2+(S\mathbf{t}_2)\mathbf{t}_1\mathrm{d}\mathbf{t}_2$$

$$\mathbf{t}_1{\triangleleft}\mathbf{s}_2=\mathbf{t}_1R,\quad(\mathrm{d}\mathbf{t}_1){\triangleleft}\mathbf{s}_2=(\mathrm{d}\mathbf{t}_1)R,\quad\mathbf{t}_1{\triangleleft}\mathrm{d}\mathbf{s}_2=0,$$

and the coaction of $\Omega(A)\bowtie_{\mathcal{R}}\Omega(A)$ on $\Omega(\underline{A})$ as a super comodule algebra is given by

$$\Delta_R\mathbf{u} = \mathbf{u}\otimes S\mathbf{s}\otimes\mathbf{t}, \quad \Delta_{R*}\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{u}\otimes S\mathbf{s}\otimes\mathbf{t} + \mathbf{u}\otimes\mathrm{d}S\mathbf{s}\otimes\mathbf{t} + \mathbf{u}\otimes S\mathbf{s}\otimes\mathrm{d}\mathbf{t},$$

where $\mathbf{u} = (u^i{}_j)$ are generators of $\underline{A}$. $\square$

## 5.3 Differentials by super double cross coproduct

Let $H$ and $A$ be two bialgebras or Hopf algebras with $A$ a right $H$-comodule algebra with right coaction $\alpha : A \to A\otimes H$ and $H$ a left $A$-comodule algebra with left coaction $\beta : H \to A\otimes H$. Suppose that $\alpha$ and $\beta$ are compatible as in [19] so that they form a double cross coproduct $H\blacktriangleright\!\!\blacktriangleleft A$.

Let $\Omega(H)$ and $\Omega(A)$ be strongly bicovariant exterior algebras, and let $\alpha$ and $\beta$ are differentiable, i.e., they extend to $\alpha_* : \Omega(A) \to \Omega(A)\underline{\otimes}\Omega(H)$ and $\beta_* : \Omega(H) \to \Omega(A)\underline{\otimes}\Omega(H)$ as super comodule algebras, which mean $\alpha_*$ and $\beta_*$ commute with $\mathrm{d}' = \mathrm{d}_A\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d}_H$. Suppose further that $\alpha_*$ and $\beta_*$ are compatible in the following sense :

$$(\Delta_{A*}\otimes\mathrm{id})\circ\alpha_*(\omega) =((\mathrm{id}\otimes\beta_*)\circ\alpha_*(\omega_{(1)}))(1\otimes\alpha_*(\omega_{(2)})) \tag{5.3.1}$$

$$(\mathrm{id}\otimes\Delta_{H*})\circ\beta_*(\eta) =(\beta_*(\eta_{(1)})\otimes\mathrm{id})((\alpha_*\otimes\mathrm{id})\circ\beta_*(\eta_{(2)})) \tag{5.3.2}$$

$$\alpha_*(\omega)\beta_*(\eta) =(-1)^{|\omega||\eta|}\beta_*(\eta)\alpha_*(\omega), \tag{5.3.3}$$

then there is a super double cross coproduct $\Omega(H)\blacktriangleright\!\!\blacktriangleleft\Omega(A)$ with super tensor product algebra structure, counit, and

$$\Delta_*(\eta\otimes\omega) = (-1)^{|\omega_{(1)}||\eta_{(2)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)}.$$

We omit the proof that $\Omega(A)\blacktriangleright\!\!\blacktriangleleft\Omega(H)$ is a super Hopf algebra since this is just a super version to the usual double cross coproduct [19].

**Theorem 5.3.1.** *Let $A, H$ be bialgebras or Hopf algebras forming a double cross coproduct $H\blacktriangleright\!\!\blacktriangleleft A$, and let $\Omega(A), \Omega(H)$ be strongly bicovariant exterior algebras with $\alpha_*, \beta_*$*

*satisfying (5.3.1)-(5.3.3). Then* $\Omega(H\bowtie A) := \Omega(H)\bowtie\Omega(A)$ *is a strongly bicovariant exterior algebra on* $H\bowtie A$ *with differential*

$$\mathrm{d}(\eta\otimes\omega)=\mathrm{d}_H\eta\otimes\omega+(-1)^{|\eta|}\eta\otimes\mathrm{d}_A\omega.$$

*Proof.* Since the algebra structure is the super tensor product, the graded Leibniz rule is already proved in Lemma 2.5.7. We need to prove d is a super-coderivation. Thus,

$$\begin{aligned}
\Delta_*\mathrm{d}(\eta\otimes\omega) &= \Delta_*(\mathrm{d}_H\eta\otimes\omega)+(-1)^{|\eta|}\Delta_*(\eta\otimes\mathrm{d}_A\omega)\\
&=(-1)^{|\omega_{(1)}||\eta_{(2)}|}\mathrm{d}_H\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)}\\
&\quad+(-1)^{|\omega_{(1)}||\eta_{(2)}|+|\omega_{(1)}|+|\eta_{(1)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\mathrm{d}_H\eta_{(2)})\otimes\omega_{(2)}\\
&\quad+(-1)^{|\omega_{(1)}||\eta_{(2)}|+|\eta_{(1)}|}\eta_{(1)}\otimes\alpha_*(\mathrm{d}_A\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)}\\
&\quad+(-1)^{|\omega_{(1)}||\eta_{(2)}|+|\eta|+|\omega_{(1)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\mathrm{d}_A\omega_{(2)}\\
&=(-1)^{|\omega_{(1)}||\eta_{(2)}|}\mathrm{d}_H\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)}\\
&\quad+(-1)^{|\omega_{(1)}||\eta_{(2)}|+|\eta_{(1)}|}\eta_{(1)}\otimes\mathrm{d}'(\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)}))\otimes\omega_{(2)}\\
&\quad+(-1)^{|\omega_{(1)}||\eta_{(2)}|+|\eta|+|\omega_{(1)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)}\\
&=\big((\mathrm{d}_H\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d}_A)\otimes\mathrm{id}\big)((-1)^{|\omega_{(1)}||\eta_{(2)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)})\\
&\quad+(-1)^{|\ |}\big(\mathrm{id}\otimes(\mathrm{d}_H\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d}_A)\big)((-1)^{|\omega_{(1)}||\eta_{(2)}|}\eta_{(1)}\otimes\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})\otimes\omega_{(2)})\\
&=(\mathrm{d}\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*(\eta\otimes\omega).
\end{aligned}$$

In the first equality we expand $\mathrm{d}(\eta\otimes\omega)$ by its definition, with its comultiplication in the second equality. Notice that $\beta_*(\mathrm{d}_H\eta_{(2)})=\mathrm{d}'(\beta_*(\eta_{(2)}))$ and $\alpha_*(\mathrm{d}_A\omega_{(1)})=\mathrm{d}'(\alpha_*(\omega_{(1)}))$, and thus we obtain the third equality by Leibniz rule of $\mathrm{d}'$ on $\alpha_*(\omega_{(1)})\beta_*(\eta_{(2)})$. Expanding $\mathrm{d}'=\mathrm{d}_A\otimes\mathrm{id}+(-1)^{|\ |}\mathrm{id}\otimes\mathrm{d}_H$, one can rewrite to obtain fourth equality, which is equivalent to the fifth equality. ☐

**Corollary 5.3.2.** *Under the condition of Theorem 5.3.1, the left coaction* $\Delta_L : H\to H\bowtie A\otimes H$ *as comodule algebra given by* $\Delta_L h=h_{(1)}\otimes\beta(h_{(2)})$ *is differentiable. Similarly, the right coaction* $\Delta_R : A\to A\otimes H\bowtie A$ *as comodule algebra given by* $\Delta_R a=\alpha(a_{(1)})\otimes a_{(2)}$

*is differentiable.*

*Proof.* Write $\alpha(a) = a^{\widetilde{(0)}} \otimes a^{\widetilde{(1)}}$ and $\beta(h) = h^{\overline{(1)}} \otimes h^{\overline{(\infty)}}$ (summations understood). We check that $\Delta_L$ is a coaction as follows

$$
\begin{aligned}
(\mathrm{id} \otimes \Delta_L)\Delta_L h =& h_{(1)} \otimes h_{(2)}{}^{\overline{(1)}} \otimes h_{(2)}{}^{\overline{(\infty)}}{}_{(1)} \otimes h_{(2)}{}^{\overline{(\infty)}}{}_{(2)}{}^{\overline{(1)}} \otimes h_{(2)}{}^{\overline{(\infty)}}{}_{(2)}{}^{\overline{(1)}} \\
=& h_{(1)} \otimes h_{(2)}{}^{\overline{(1)}} h_{(3)}{}^{\overline{(1)}\widetilde{(0)}} \otimes h_{(2)}{}^{\overline{(\infty)}} h_{(3)}{}^{\overline{(1)}\widetilde{(1)}} \otimes h_{(3)}{}^{\overline{(\infty)}\overline{(1)}} \otimes h_{(3)}{}^{\overline{(\infty)}\overline{(\infty)}} \\
=& h_{(1)} \otimes h_{(2)}{}^{\overline{(1)}} h_{(3)}{}^{\overline{(1)}}{}_{(1)}{}^{\widetilde{(0)}} \otimes h_{(2)}{}^{\overline{(\infty)}} h_{(3)}{}^{\overline{(1)}}{}_{(1)}{}^{\widetilde{(1)}} \otimes h_{(3)}{}^{\overline{(1)}}{}_{(2)} \otimes h_{(3)}{}^{\overline{(\infty)}} \\
=& h_{(1)} \otimes \beta(h_{(2)})\alpha(h_{(3)}{}^{\overline{(1)}}{}_{(1)}) \otimes h_{(3)}{}^{\overline{(1)}}{}_{(2)} \otimes h_{(3)}{}^{\overline{(\infty)}} \\
=& h_{(1)} \otimes \alpha(h_{(3)}{}^{\overline{(1)}}{}_{(1)})\beta(h_{(2)}) \otimes h_{(3)}{}^{\overline{(1)}}{}_{(2)} \otimes h_{(3)}{}^{\overline{(\infty)}} \\
=& \Delta(h_{(1)} \otimes h_{(2)}{}^{\overline{(1)}}) \otimes h_{(2)}{}^{\overline{(\infty)}} = (\Delta \otimes \mathrm{id})\Delta_L h.
\end{aligned}
$$

We apply the definition of $\Delta_L$ twice in the first equality, and we use the condition (5.3.2) on $h_{(2)}{}^{\overline{(\infty)}}$ in our notation of $\alpha$ and $\beta$ to obtain the second equality. Then since $\beta$ is a coaction, we rewrite $(\mathrm{id} \otimes \beta)\beta(h_{(3)}{}^{\overline{(\infty)}}) = (\Delta_H \otimes \mathrm{id})\beta(h_{(3)}{}^{\overline{(\infty)}})$ and obtain the third equality, which equivalent to the fourth equality. Then we use condition (5.3.3) to get the fifth equality, which is equal to the sixth equality. We also check that $\Delta_L$ is an algebra map:

$$
\begin{aligned}
\Delta_L(hg) =& (h_{(1)}g_{(1)}) \otimes \beta(h_{(2)}g_{(2)}) = h_{(1)}g_{(1)} \otimes \beta(h_{(2)})\beta(g_{(2)}) \\
=& (h_{(1)} \otimes \beta(h_{(2)}))(g_{(1)} \otimes \beta(g_{(2)})) = (\Delta_L h)(\Delta_L g).
\end{aligned}
$$

Thus, $H$ is a left $H{\bowtie}A$-comodule algebra. Similarly for $\Delta_R$ a coaction,

$$
\begin{aligned}
(\Delta_R \otimes \mathrm{id})\Delta_R a =& a_{(1)}{}^{\widetilde{(0)}}{}_{(1)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(0)}}{}_{(1)}{}^{\widetilde{(1)}} \otimes a_{(1)}{}^{\widetilde{(0)}}{}_{(2)} \otimes a_{(1)}{}^{\widetilde{(0)}} \otimes a_{(2)} \\
=& a_{(1)}{}^{\widetilde{(0)}\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(0)}\widetilde{(1)}} \otimes a_{(1)}{}^{\widetilde{(1)}\overline{(1)}} a_{(2)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(1)}\overline{(\infty)}} a_{(2)}{}^{\widetilde{(1)}} \otimes a_{(3)} \\
=& a_{(1)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(1)}}{}_{(1)} \otimes a_{(1)}{}^{\widetilde{(1)}}{}_{(2)}{}^{\overline{(1)}} a_{(2)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(1)}}{}_{(2)}{}^{\overline{(\infty)}} a_{(2)}{}^{\widetilde{(1)}} \otimes a_{(3)} \\
=& a_{(1)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(1)}}{}_{(1)} \otimes \beta(a_{(1)}{}^{\widetilde{(1)}}{}_{(2)})\alpha(a_{(2)}) \otimes a_{(3)} \\
=& a_{(1)}{}^{\widetilde{(0)}} \otimes a_{(1)}{}^{\widetilde{(1)}}{}_{(1)} \otimes \alpha(a_{(2)})\beta(a_{(1)}{}^{\widetilde{(1)}}{}_{(2)}) \otimes a_{(3)}
\end{aligned}
$$

$$=a_{(1)}{}^{\widetilde{(0)}} \otimes \Delta(a_{(1)}{}^{\widetilde{(1)}} \otimes a_{(2)}) = (\mathrm{id} \otimes \Delta)\Delta_R a$$

and is again an algebra map. Since $\beta_*$ and $\alpha_*$ globally exist as assumed in Theorem 5.3.1, it is clear that we can define

$$\Delta_{L*}\eta = \eta_{(1)} \otimes \beta_*(\eta_{(2)}), \quad \Delta_{R*}\omega = \alpha_*(\omega_{(1)}) \otimes \omega_{(2)}$$

for all $\eta \in \Omega(H)$ and $\omega \in \Omega(A)$, and that they have the required properties by our assumptions on $\alpha_*, \beta_*$ and a super version of the above proofs for $\Delta_L$ and $\Delta_R$. For example on degree 1, we have

$$\Delta_{L*}\mathrm{d}_H h = \mathrm{d}_H h_{(1)} \otimes \beta(h_{(2)}) + h_{(1)} \otimes \beta_*(\mathrm{d}_H h_{(2)}),$$

$$\Delta_{R*}\mathrm{d}_A a = \alpha_*(\mathrm{d}a_{(1)}) \otimes a_{(2)} + \alpha(a_{(1)}) \otimes \mathrm{d}_A a_{(2)}.$$

$\square$

As an application, we find in principle a strongly bicovariant exterior algebra of the quantum codouble $coD(U_q(su_2)) = U_q(su_2)^{\mathrm{cop}} \blacktriangleright\!\!\blacktriangleleft \mathbb{C}_q[SU_2]$ as a version of $\mathbb{C}_q[SO_{1,3}]$ such that $coD(U_q(su_2))$ coacts differentiably on $U_q(su_2)$, viewed as a version of unit "sphere" in $q$-Minkowski space [34]. We omit the detail as it is essentially equivalent to the case of $\mathbb{C}_q[SU_2] \bowtie_{\mathcal{R}} \mathbb{C}[SU_2]$ in Example 5.2.9.

# Chapter 6

# Exterior algebra on quantum groups obtained by bosonisation

Let $A$ be a coquasitriangular Hopf algebra with a strongly bicovariant exterior algebra $\Omega(A)$, and let $\underline{A}$ be the transmutation of $A$. In Section 6.1, we show that the known isomorphism $A \bowtie_{\mathcal{R}} A \cong A \rtimes \underline{A}$ in [19] extends to $\Omega(A) \bowtie_{\mathcal{R}} \Omega(A) \cong \Omega(A) \rtimes \Omega(\underline{A})$ as exterior algebras, and the latter gives a strongly bicovariant exterior algebra on bosonisation $A \rtimes \underline{A}$. This motivates a general construction of a super bosonisation $\Omega(A) \rtimes \Omega(B)$, where $\Omega(B)$ is a strongly bicovariant exterior algebra on braided Hopf algebra $B$. This is given in Section 6.2 and we will show that $\Omega(A) \rtimes \Omega(B)$ gives a natural exterior algebra on $A \rtimes B$ such that the canonical coaction $\Delta_R : B \to B \otimes A \rtimes B$ extends to $\Delta_{R*} : \Omega(B) \to \Omega(B) \underline{\otimes} \Omega(A \rtimes B)$ differentiably.

As an application, in Section 6.4 we recover the natural differential calculus of the Sweedler-Taft algebra $U_q(b_+)$ in the known classification[41], but we think of it as a $q$-deformed coordinate algebra $\mathbb{C}_q[B_+] = \mathbb{C}[t, t^{-1}] \rtimes \mathbb{C}[x]$, where $\mathbb{C}[x]$ is a braided-line in the category of $\mathbb{Z}$-graded spaces, such that $\mathbb{C}_q[B_+]$ coacts on $\mathbb{C}[x]$ differentiably. We also find the natural strongly bicovariant exterior algebra of $\mathbb{C}_q[P] = \mathbb{C}_q[GL_2] \rtimes \mathbb{C}_q^2$ as a quantum deformation of a maximal parabolic subgroup $P \subset SL_3$, where $\mathbb{C}_q^2$ is a two-dimensional quantum-braided plane in the category of $\mathbb{C}_q[GL_2]$. These examples are part of a more general construction of $\Omega(A{\triangleright\!\!\!<}V(R)) = \Omega(A){\triangleright\!\!\!<}\Omega(V(R))$ in Section 6.3. Here $A = A(R)[D^{-1}]$ is a Hopf algebra localisation of the FRT bialgebra $A(R)$[12] with invertible quantum determinant $D$ which is assumed grouplike and central in $A(R)$, and $V(R)$ is an additive braided plane in the category of $A$-crossed modules.

## 6.1 The special case of transmutation

Given a coquasitriangular Hopf algebra $(A, \mathcal{R})$, its *transmutation* is a braided-Hopf algebra $\underline{A}$ introduced in [24] as one of the main constructions in the theory of braided Hopf algebra (the other one is bosonisation as discussed in Section 2.4). Here $\underline{A}$ has the same coalgebra structure as $A$ but a modified product $a \bullet b = a_{(2)}b_{(2)}\mathcal{R}((Sa_{(1)})a_{(3)}, Sb_{(1)})$, and lives in the braided category of right-modules $\mathcal{M}^A$ where the right coaction is given by the adjoint coaction $\mathrm{Ad}_R a = a_{(2)} \otimes (Sa_{(1)})a_{(3)}$. Furthermore, there is a right action of $A$ on $\underline{A}$ given by $b \triangleleft a = b^{\overline{(0)}}\mathcal{R}(b^{\overline{(1)}}, a) = b_{(2)}\mathcal{R}((Sb_{(1)})b_{(3)}, a)$ making $\underline{A} \in \mathcal{M}^A_A$ (this follows from a functor $\underline{A} \in \mathcal{M}^A \hookrightarrow \mathcal{M}^A_A$, see Section 2.4). Either by cobosonisation or from the crossed module point of view, one has an ordinary Hopf algebra $A{\triangleright\!\!\!<}\underline{A}$. It is known [19, Theorem 7.4.10] that $A{\triangleright\!\!\!<}\underline{A} \cong A{\bowtie}_{\mathcal{R}}A$. Similarly, regarding a strongly bicovariant $\Omega(A)$ as coquastriangular by extending $\mathcal{R}$ by zero, it transmutes to a super-braided Hopf algebra $\underline{\Omega(A)}$. We write $\Omega(A) \cong A{\triangleright\!\!\!<}\Lambda$ as generated by $A, \Lambda^1$ as discussed above.

**Lemma 6.1.1.** *[34, Proposition 8] If $\Omega(A)$ is a strongly bicovariant exterior algebra on a coquasitriangular Hopf algebra $A$ then $\Omega(\underline{A}) := \underline{\Omega(A)}$ is a braided exterior algebra on $\underline{A}$ generated by $\underline{A}$ and $\Lambda^1$ with relations*

$$a \bullet v = av, \quad v \bullet a = a_{(2)} \bullet (v^{\overline{(0)}} \triangleleft a_{(3)})\mathcal{R}(v^{\overline{(1)}}, Sa_{(1)}), \quad v \bullet w = vw$$

*for all $v, w \in \Lambda^1$ and $a \in \underline{A}$, and the differential map of $\underline{\Omega(A)}$ is the same as the differential map of $\Omega(A)$.*

By construction, $\Omega(\underline{A}) := \underline{\Omega(A)}$ is a super braided Hopf algebra in $\mathcal{M}^{\Omega(A)}_{\Omega(A)}$ by adjoint coaction and the induced action. Thus, we have super bosonisation $\Omega(A){\triangleright\!\!<}\Omega(\underline{A})$ as in [19] but now with signs,

$$
\begin{aligned}
(\omega\otimes\eta)(\tau\otimes\xi) &= (-1)^{|\eta||\tau|}\omega\tau_{(1)}\otimes\eta_{(2)}\xi_{(2)}\mathcal{R}((S\eta_{(1)})\eta_{(3)},\tau_{(2)}S\xi_{(1)})\\
\Delta_*(\omega\otimes\eta) &= (-1)^{(|\omega_{(2)}|+|\eta_{(1)}|)|\eta_{(2)}|}\omega_{(1)}\otimes\eta_{(2)}\otimes\omega_{(2)}(S\eta_{(1)})\eta_{(3)}\otimes\eta_{(4)}
\end{aligned}
$$

for all $\omega,\tau\in\Omega(A)$ and $\eta,\xi\in\Omega(\underline{A})$.

**Proposition 6.1.2.** *Let $\Omega(A)$ be a strongly bicovariant exterior algebra on a coquasitriangular Hopf algebra $A$ and $\underline{A}$ the transmutation of $A$. Then $\Omega(A{\triangleright\!\!<}\underline{A}):=\Omega(A){\triangleright\!\!<}\underline{\Omega(A)}$ is a strongly bicovariant exterior algebra on $A{\triangleright\!\!<}\underline{A}$. Moreover, $\Omega(A{\triangleright\!\!<}\underline{A})\cong\Omega(A{\bowtie_{\mathcal{R}}}A)$ as an isomorphism of differential exterior algebras.*

*Proof.* For the first part, we define d to be the graded sum of $\mathrm{d}_A$ and $\mathrm{d}_{\underline{A}}$ as in Corrolary 5.2.8. That this gives a strongly bicovariant exterior algebra is a special case of Theorem 6.2.1 proven later so we omit the details. It is also clear once we have proven the asserted isomorphism. For the latter, let $\varphi: A{\bowtie_{\mathcal{R}}}A\to A{\triangleright\!\!<}\underline{A}, \varphi(a\otimes b)=ab_{(1)}\otimes b_{(2)}$ which is a Hopf algebra isomorphism by [19, Theorem 7.4.10]. We extend this to a map $\varphi_*:\Omega(A){\bowtie_{\mathcal{R}}}\Omega(A)\to\Omega(A){\triangleright\!\!<}\underline{\Omega(A)}$ by $\varphi_*(\omega\otimes\eta)=\omega\eta_{(1)}\otimes\eta_{(2)}$ for all $\omega,\eta\in\Omega(A)$. It is straightforward to check that $\varphi_*$ is a super-Hopf algebra map. We also need to check that $\varphi_*$ commutes with d and note that d on both sides is the graded sum of the d on each tensor factor. We check

$$
\begin{aligned}
\mathrm{d}(\varphi_*(\omega\otimes\eta)) =&\mathrm{d}(\omega\eta_{(1)}\otimes\eta_{(2)})=\mathrm{d}_A(\omega\eta_{(1)})\otimes\eta_{(2)}+(-1)^{|\omega\eta_{(1)}|}\omega\eta_{(1)}\otimes\mathrm{d}_A\eta_{(2)}\\
=&(\mathrm{d}_A\omega)\eta_{(1)}\otimes\eta_{(2)}+(-1)^{|\omega|}\omega(\mathrm{d}_A\eta_{(1)})\otimes\eta_{(2)}+(-1)^{|\omega|+|\eta_{(1)}|}\omega\eta_{(1)}\otimes\mathrm{d}_A\eta_{(2)}\\
=&(\mathrm{d}_A\omega)\eta_{(1)}\otimes\eta_{(2)}+(-1)^{|\omega|}\omega(\mathrm{d}_A\eta)_{(1)}\otimes(\mathrm{d}_A\eta)_{(2)}\\
=&\varphi_*(\mathrm{d}_A\omega\otimes\eta+(-1)^{|\omega|}\omega\otimes\mathrm{d}_A\eta)=\varphi_*(\mathrm{d}(\omega\otimes\eta)).
\end{aligned}
$$

The map $\varphi_*$ is invertible with inverse $\phi_*(\omega\otimes\eta)=\omega S\eta_{(1)}\otimes\eta_{(2)}$. ∎

## 6.2 Differentials by super bosonisation

We generalise the first part of Proposition 6.1.2 replacing $A$ by any Hopf algebra with and $\underline{A}$ by any braided Hopf algebra $B$ in the category of right $A$-crossed modules.

Thus, let $A$ be a Hopf algebra with strongly bicovariant exterior algebra $\Omega(A)$. Let $B$ be a right $A$-comodule algebra with coaction $\Delta_R b = b^{\overline{(0)}} \otimes b^{\overline{(1)}}$ such that $\Delta_R$ extends to $\Delta_{R*} : \Omega(B) \to \Omega(B)\underline{\otimes}\Omega(A)$ by $\Delta_{R*}\eta = \eta^{\overline{(0)}*} \otimes \eta^{\overline{(1)}*}$ as differentiable coaction.

We now come to the genuinely new data: (i) assume that $A$ also acts on $\Omega(B)$ so as to make this an $A$-crossed module algebra (an algebra in $\mathcal{M}^A_A$); (ii) suppose that this action extends to $\triangleleft : \Omega(B)\underline{\otimes}\Omega(A) \to \Omega(B)$ as differentiable action; (iii) suppose that $B$ is a braided Hopf algebra in $\mathcal{M}^A_A$; (iv) suppose that $\Omega(B)$ is *braided strongly bicovariant* in the sense that $\underline{\Delta}$ extends to a degree preserving super braided coproduct $\underline{\Delta}_*$ making $\Omega(B)$ a super braided Hopf algebra in $\mathcal{M}^{\Omega(A)}_{\Omega(A)}$ and $\mathrm{d}_B$ a super-coderivation in the sense of (2.5.1).

Also, given a super-braided Hopf algebra $\Omega(B) \in \mathcal{M}^{\Omega(A)}_{\Omega(A)}$, the usual bosonisation formulae extend with signs to define a super bosonisation $\Omega(A){\triangleright\!\!\!<}\Omega(B)$ with

$$(\omega \otimes \eta)(\tau \otimes \xi) = (-1)^{|\eta||\tau_{(1)}|}\omega\tau_{(1)} \otimes (\eta\triangleleft\tau_{(2)})\xi$$

$$\Delta_*(\omega \otimes \eta) = (-1)^{|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(1)} \otimes \eta_{\underline{(1)}}{}^{\overline{(0)}*} \otimes \omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*} \otimes \eta_{\underline{(2)}}$$

for all $\omega, \tau \in \Omega(A)$ and $\eta, \xi \in \Omega(B)$.

**Theorem 6.2.1.** *Let $A$ be a Hopf algebra and let $B$ be a braided Hopf algebra in $\mathcal{M}^A_A$, $\Omega(A)$ a strongly bicovariant exterior algebra and $\Omega(B)$ a braided strongly bicovariant exterior algebra in $\mathcal{M}^{\Omega(A)}_{\Omega(A)}$ (so the action and coaction of $A$ is differentiable and $\mathrm{d}_B$ is a super coderivation). Then $\Omega(A{\triangleright\!\!\!<}B) := \Omega(A){\triangleright\!\!\!<}\Omega(B)$ is a strongly bicovariant exterior algebra on $A{\triangleright\!\!\!<}B$ with differential*

$$\mathrm{d}(\omega \otimes \eta) = \mathrm{d}_A\omega \otimes \eta + (-1)^{|\omega|}\omega \otimes \mathrm{d}_B\eta$$

*for all* $\omega \in \Omega(A)$, $\eta \in \Omega(B)$.

*Proof.* First note that $\Omega^1(A{\triangleright\!\!\!<}B) = \mathrm{span}\{(a\otimes b)\mathrm{d}(c\otimes d)\} = \mathrm{span}\{a\mathrm{d}_A c\otimes b + a'\otimes b'\mathrm{d}_B d'\} = \Omega^1(A)\otimes B \oplus A{\triangleright\!\!\!<}\Omega^1(B)$ since

$$(a\otimes 1)\mathrm{d}(c\otimes b) - (ac\otimes 1)\mathrm{d}(1\otimes b) = a\mathrm{d}_A c\otimes b, \quad (a'\otimes b')\mathrm{d}(1\otimes d') = a'\otimes b'\mathrm{d}_B d'$$

for all $a', a, c \in A$ and $b, b', d' \in B$. The graded Leibniz rule holds since

$$\begin{aligned}
\mathrm{d}(\eta\omega) \equiv& \mathrm{d}((1\otimes\eta)(\omega\otimes 1))\\
=&(-1)^{|\eta||\omega_{(1)}|}\big((-1)^{|\omega_{(1)}|}\omega_{(1)}\otimes(\mathrm{d}_B\eta)\omega_{(2)} + \mathrm{d}_A\omega_{(1)}\otimes\eta{\triangleleft}\omega_{(2)} + (-1)^{|\eta|+|\omega_{(1)}|}\omega\otimes\eta{\triangleleft}\mathrm{d}_A\omega_{(2)}\big)\\
=&(1\otimes\mathrm{d}_B\eta)(\omega\otimes 1) + (-1)^{|\eta|}(1\otimes\eta)(\mathrm{d}_A\omega\otimes 1) \equiv (\mathrm{d}\eta)\omega + (-1)^{|\eta|}\eta\mathrm{d}\omega
\end{aligned}$$

for all $\eta \in \Omega(B)$ and $\omega \in \Omega(A)$. Clearly $\mathrm{d}^2 = 0$ and thus $\Omega(A){\triangleright\!\!\!<}\Omega(B)$ is a DGA. We also show that the calculus on $A{\triangleright\!\!\!<}B$ is strongly bicovariant by showing that d is a super-coderivation as follow

$$\begin{aligned}
\Delta_*\mathrm{d}(\omega\eta) =&(\Delta_*\mathrm{d}\omega)\Delta_*(\eta) + (-1)^{|\omega|}\Delta_*(\omega)\Delta_*(\mathrm{d}\eta)\\
=&(-1)^{|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}(\mathrm{d}\omega_{(1)})\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}}\\
&+(-1)^{|\omega_{(1)}|+|\mathrm{d}\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(1)}\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes(\mathrm{d}\omega_{(2)})\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}}\\
&+(-1)^{|\omega|+|\omega_{(2)}||\mathrm{d}\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(1)}\mathrm{d}\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}}\\
&+(-1)^{|\omega|+|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|+|\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(1)}\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes\omega_{(2)}\mathrm{d}(\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}})\\
=&(-1)^{|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\mathrm{d}(\omega_{(1)}\eta_{\underline{(1)}}{}^{\overline{(0)}*})\otimes\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}}\\
&+(-1)^{|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|+|\omega_{(1)}\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes\mathrm{d}(\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}})\\
=&(\mathrm{d}\otimes\mathrm{id} + (-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})((-1)^{|\omega_{(2)}||\eta_{\underline{(1)}}{}^{\overline{(0)}*}|}\omega_{(1)}\eta_{\underline{(1)}}{}^{\overline{(0)}*}\otimes\omega_{(2)}\eta_{\underline{(1)}}{}^{\overline{(1)}*}\eta_{\underline{(2)}})\\
=&(\mathrm{d}\otimes\mathrm{id} + (-1)^{|\ |}\mathrm{id}\otimes\mathrm{d})\Delta_*(\omega\eta),
\end{aligned}$$

which completes the proof. □

In practice since $\Omega(A)$ and $\Omega(B)$ are generated by their elements of degree 0 and 1, we can construct the bosonisation $\Omega(A){\triangleright\!\!\!<}\Omega(B)$ providing that we know the action and coaction on degree 0 and degree 1.

**Lemma 6.2.2.** *(i) If $B$ is an $A$-crossed module with differentiable action and coaction obeying*

$$\Delta_{R*}((\mathrm{d}_B b)\triangleleft a) = (\mathrm{d}_B b^{\overline{(0)}})\triangleleft a_{(2)} \otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)} + b^{\overline{(0)}}\triangleleft a_{(2)} \otimes Sa_{(1)}(\mathrm{d}_A b^{\overline{(1)}})a_{(3)}$$

*for all $b \in B$ and for all $a \in A$, then $\Omega(B)$ is a super $\Omega(A)$-crossed module algebra.*

*(ii) Furthermore, if $B$ is a braided Hopf algebra in $\mathcal{M}_A^A$,*

$$\underline{\Delta}(b\mathrm{d}_B c) = b_{\underline{(1)}}\mathrm{d}_B c_{\underline{(1)}}{}^{\overline{(0)}}\otimes(b_{\underline{(2)}}\triangleleft c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}}+b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}}\otimes\big((b_{\underline{(2)}}\triangleleft\mathrm{d}_A c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}}+(b_{\underline{(2)}}\triangleleft c_{\underline{(1)}}{}^{\overline{(1)}})\mathrm{d}_B c_{\underline{(2)}}\big)$$

*is well-defined, and $\Omega(B)$ is the maximal prolongation of $\Omega^1(B)$, then $\Omega(B)$ is a braided strongly bicovariant calculus, i.e. a super braided Hopf algebra in $\mathcal{M}_{\Omega(A)}^{\Omega(A)}$ such that $\mathrm{d}_B$ is a super-coderivation in the sense of (2.5.1).*

*Proof.* (i) We first check,

$$\begin{aligned}\Delta_{R*}(b\triangleleft\mathrm{d}_A a) =&\Delta_{R*}(\mathrm{d}_B(b\triangleleft a) - (\mathrm{d}_B b)\triangleleft a)\\
=&\mathrm{d}_B(b\triangleleft a)^{\overline{(0)}} \otimes (b\triangleleft a)^{\overline{(1)}} + (b\triangleleft a)^{\overline{(0)}} \otimes \mathrm{d}_A(b\triangleleft a)^{\overline{(1)}} - \Delta_{R*}((\mathrm{d}_B\eta)\triangleleft a)\\
=&\mathrm{d}_B(b^{\overline{(0)}}\triangleleft a_{(2)}) \otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)} + b^{\overline{(0)}}\triangleleft a_{(2)} \otimes \mathrm{d}_A((Sa_{(1)})b^{\overline{(1)}}a_{(3)})\\
&- (\mathrm{d}_B b^{\overline{(0)}})\triangleleft a_{(2)} \otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)} - b^{\overline{(0)}}\triangleleft a_{(2)} \otimes Sa_{(1)}(\mathrm{d}_A b^{\overline{(1)}})a_{(3)}\\
=&b^{\overline{(0)}}\triangleleft\mathrm{d}_A a_{(2)} \otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)} + b^{\overline{(0)}}\triangleleft a_{(2)} \otimes (\mathrm{d}_A Sa_{(1)})b^{\overline{(1)}}a_{(3)}\\
&+ b^{\overline{(0)}}\triangleleft a_{(2)} \otimes (Sa_{(1)})b^{\overline{(1)}}\mathrm{d}_A a_{(3)}\\
=&b^{\overline{(0)}}\triangleleft(\mathrm{d}_A a)_{(2)} \otimes S(\mathrm{d}_A a)_{(1)}b^{\overline{(1)}}(\mathrm{d}_A a)_{(3)}.\end{aligned}$$

This makes $B$ is a crossed module as regards the action and coaction of $\Omega^1(A)$. Similarly for $\Omega^1(B)$, where we check using (5.1.1),

$$
\begin{aligned}
\Delta_{R*}((\mathrm{d}_B b)\triangleleft \mathrm{d}_A a) = \Delta_{R*}(\mathrm{d}_B(b\triangleleft \mathrm{d}_A a))&\\
=&\mathrm{d}_B(b\triangleleft \mathrm{d}_A a)^{\overline{(0)}}\otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)}+(-1)^{|(b\triangleleft \mathrm{d}_A a)^{\overline{(0)}}|}(b\triangleleft \mathrm{d}_A a)^{\overline{(0)}}\otimes \mathrm{d}_A(b\triangleleft \mathrm{d}_A a)^{\overline{(1)}}\\
=&\mathrm{d}_B(b^{\overline{(0)}}\triangleleft \mathrm{d}_A a_{(2)})\otimes (Sa_{(1)})b^{\overline{(0)}}a_{(3)}+\mathrm{d}_B(b^{\overline{(0)}}\triangleleft a_{(2)})\otimes(\mathrm{d}_A Sa_{(1)})b^{\overline{(1)}}a_{(3)}\\
&+\mathrm{d}_B(b^{\overline{(0)}}\triangleleft a_{(2)})\otimes (Sa_{(1)})b^{\overline{(1)}}\mathrm{d}_A a_{(3)}-b^{\overline{(0)}}\triangleleft \mathrm{d}_A a_{(2)}\otimes \mathrm{d}_A((Sa_{(1)})b^{\overline{(1)}}a_{(3)})\\
&+b^{\overline{(0)}}\triangleleft a_{(2)}\otimes \mathrm{d}_A((\mathrm{d}_A Sa_{(1)})b^{\overline{(1)}}a_{(3)})+b^{\overline{(0)}}\triangleleft a_{(2)}\otimes \mathrm{d}_A(Sa_{(1)}b^{\overline{(1)}}\mathrm{d}_A a_{(3)})\\
=&(\mathrm{d}_B b^{\overline{(0)}})\triangleleft \mathrm{d}_A a_{(2)}\otimes (Sa_{(1)})b^{\overline{(1)}}a_{(3)}+(\mathrm{d}_B b^{\overline{(0)}})\triangleleft a_{(2)}\otimes (S\mathrm{d}_A a_{(1)})b^{\overline{(1)}}a_{(3)}\\
&+(\mathrm{d}_B b^{\overline{(0)}})\triangleleft a_{(2)}\otimes (Sa_{(1)})b^{\overline{(1)}}\mathrm{d}_A a_{(3)}-b^{\overline{(0)}}\triangleleft \mathrm{d}_A a_{(2)}\otimes (Sa_{(1)})(\mathrm{d}_A b^{\overline{(1)}})a_{(3)}\\
&-b^{\overline{(0)}}\triangleleft a_{(2)}\otimes (S\mathrm{d}_A a_{(1)})(\mathrm{d}_A b^{\overline{(1)}})a_{(3)}+b^{\overline{(0)}}\triangleleft a_{(2)}\otimes (Sa_{(1)})(\mathrm{d}_A b^{\overline{(1)}})a_{(3)}\\
=&(\mathrm{d}_B b^{\overline{(0)}})\triangleleft (\mathrm{d}_A a)_{(2)}\otimes S(\mathrm{d}_A a)_{(1)}b^{\overline{(1)}}(\mathrm{d}_A a)_{(3)}\\
&+(-1)^{|(\mathrm{d}_A a)_{(1)}|+|(\mathrm{d}_A a)_{(2)}|}b^{\overline{(0)}}\triangleleft (\mathrm{d}_A a)_{(2)}\otimes S(\mathrm{d}_A a)_{(1)}(\mathrm{d}_B b^{\overline{(1)}})(\mathrm{d}_A a)_{(3)}\\
=&(-1)^{|(\mathrm{d}_B b)^{\overline{(1)}}|(|(\mathrm{d}_A a)_{(1)}|+|(\mathrm{d}_A a)_{(2)}|)}(\mathrm{d}_B b)^{\overline{(0)}}\triangleleft (\mathrm{d}_A a)_{(2)}\otimes S(\mathrm{d}_A a)_{(1)}(\mathrm{d}_B b)^{\overline{(1)}}(\mathrm{d}_A a)_{(3)}.
\end{aligned}
$$

Since the action and coaction of $A$ on $B$ are differentiable, and both exterior algebras are generated by degrees 0,1, it follows that $\Delta_{R*}(\eta\triangleleft\omega)$ obeys the crossed module condition in general, making $\Omega(B)$ a super $\Omega(A)$-crossed module.

(ii) We need to prove that $\underline{\Delta}_*(\xi\eta)=(\underline{\Delta}_*\xi)(\underline{\Delta}_*\eta)$, and it suffices to prove that $\underline{\Delta}_*$ extends to $\Omega^2(B)$ since $\Omega(B)$ is the maximal prolongation of $\Omega^1(B)$. Applying $\underline{\Delta}_*$ to $b\mathrm{d}_B c=0$, we have

$$b_{\underline{(1)}}\mathrm{d}_B c_{\underline{(1)}}{}^{\overline{(0)}}\otimes(b_{\underline{(2)}}\triangleleft c_{\underline{(1)}}{}^{\overline{(0)}})c_{\underline{(2)}}=0,\quad b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}}\otimes((b_{\underline{(2)}}\triangleleft \mathrm{d}_A c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}}+(b_{\underline{(2)}}\triangleleft c_{\underline{(1)}}{}^{\overline{(1)}})\mathrm{d}_B c_{\underline{(2)}})=0.$$

Applying $\mathrm{d}_B\otimes\mathrm{id}$ to the first equation, we have

$$\mathrm{d}_B b_{\underline{(1)}}\mathrm{d}_B c_{\underline{(1)}}{}^{\overline{(0)}}\otimes(b_{\underline{(2)}}\triangleleft c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}}=0$$

which is the $\Omega^2(B)\otimes B$ part of $\underline{\Delta}_*(\mathrm{d}_B b\mathrm{d}_B c)$. Applying $\mathrm{id}\otimes\mathrm{d}_B$ to the second equation, we have

$$b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}} \otimes ((\mathrm{d}_B b_{\underline{(2)}}){\triangleleft}\mathrm{d}_A c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}} + b_{\underline{(1)}}c_{\underline{(1)}}{}^{\overline{(0)}} \otimes ((\mathrm{d}_B b_{\underline{(2)}}){\triangleleft}c_{\underline{(1)}}{}^{\overline{(1)}})\mathrm{d}_B c_{\underline{(2)}} = 0$$

which is the $B \otimes \Omega^2(B)$ part of $\underline{\Delta}_*(\mathrm{d}_B b \mathrm{d}_B c)$. Finally, applying $\mathrm{d}_B \otimes \mathrm{id}$ to the second equation and $\mathrm{id} \otimes \mathrm{d}_B$ to the first equation and subtracting them, we have

$$(\mathrm{d}_B b_{\underline{(1)}})c_{\underline{(1)}}{}^{\overline{(0)}} \otimes \big((b_{\underline{(2)}}{\triangleleft}\mathrm{d}_A c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}} + (b_{\underline{(2)}}{\triangleleft}c_{\underline{(1)}}{}^{\overline{(1)}})\mathrm{d}_B c_{\underline{(2)}}\big) - b_{\underline{(1)}}\mathrm{d}_B c_{\underline{(1)}}{}^{\overline{(0)}} \otimes ((\mathrm{d}_B b_{\underline{(2)}}){\triangleleft}c_{\underline{(1)}}{}^{\overline{(1)}})c_{\underline{(2)}} = 0$$

which is the $\Omega^1(B)\underline{\otimes}\Omega^1(B)$ part of $\underline{\Delta}_*(\mathrm{d}_B b \mathrm{d}_B c)$ and thus completes the proof.

□

This lemma assists with the data needed for Theorem 6.2.1. Finally, we note that $B$ is canonically a $A{\triangleright\!\!\!<}B$-comodule algebra by $\Delta_R b = b_{\underline{(1)}}{}^{\overline{(0)}} \otimes b_{\underline{(1)}}{}^{\overline{(1)}} \otimes b_{\underline{(2)}}$.

**Corollary 6.2.3.** *Under the condition of Theorem 6.2.1, $\Delta_R : B \to B \otimes A{\triangleright\!\!\!<}B$ is differentiable.*

*Proof.* Since $\Delta_{R*} : \Omega(B) \to \Omega(B) \otimes \Omega(A)$ and $\underline{\Delta}_*$ globally exist as assumed in Theorem 6.2.1, it is clear that $\Delta_{R*}\eta = \eta_{\underline{(1)}}{}^{\overline{(0)}*} \otimes \eta_{\underline{(1)}}{}^{\overline{(1)}*} \otimes \eta_{\underline{(2)}}$ is well-defined and gives a coaction of $\Omega(A{\triangleright\!\!\!<}B)$ on $\Omega(B)$. For example, on degree 1 we have

$$\Delta_{R*}(\mathrm{d}_B b) = \mathrm{d}_B b_{\underline{(1)}}{}^{\overline{(0)}} \otimes b_{\underline{(1)}}{}^{\overline{(1)}} \otimes b_{\underline{(2)}} + b_{\underline{(1)}}{}^{\overline{(0)}} \otimes \mathrm{d}_A b_{\underline{(1)}}{}^{\overline{(1)}} \otimes b_{\underline{(2)}} + b_{\underline{(1)}}{}^{\overline{(0)}} \otimes b_{\underline{(1)}}{}^{\overline{(1)}} \otimes \mathrm{d}_B b_{\underline{(2)}}.$$

□

**Remark 6.2.4.** In the special case of $A{\triangleright\!\!\!<}\underline{A} \cong A{\bowtie}_{\mathcal{R}}A$ in Proposition 6.1.2, the latter coacts on $\underline{A}$ differentiably and we recover Corollary 5.2.8.

We will give some $q$-deformed examples of the preceding section and for this it is convenient to use $R$-matrix methods starting with the FRT bialgebra $A(R)$ and its strongly bicovariant exterior algebra. We start some results at this level for $R$ a $q$-Hecke solution of the braid relations, before moving to specific examples in following subsections. We use notations and conventions of [19].

## 6.3 Exterior algebra on $A{\triangleright\!\!\!<}V(R)$

It has been mentioned in Example 5.2.9 that the FRT bialgebra $A(R)$ is generated by $\mathbf{t} = (t^i{}_j)$ such that

$$R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R, \quad \Delta\mathbf{t} = \mathbf{t}\otimes\mathbf{t}, \quad \epsilon\mathbf{t} = \mathrm{id}$$

where $R = (R^i{}_j{}^k{}_l) \in M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ where $(i,j)$ label the first copy and $(k,l)$ the second. In the compact notation the numerical suffices indicate the position in the tensor matrix product, e.g. $\mathbf{t}_1 = \mathbf{t}\otimes\mathrm{id}$, $\mathbf{t}_2 = \mathrm{id}\otimes\mathbf{t}$, $R_{23} = \mathrm{id}\otimes R$ etc. We ask for $R$ to satisfy the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (equivalent to the braid relation) and for the present purpose to be *$q$-Hecke*, which means that it satisfies

$$(PR - q)(PR + q^{-1}) = 0, \tag{6.3.1}$$

where $P = (P^i{}_j{}^k{}_l)$ is a permutation matrix with $P^i{}_j{}^k{}_l = \delta^i{}_l\delta^k{}_j$ in terms of the Kronecker delta or identity matrix. The $q$-Hecke condition is equivalent to $R_{21}R = \mathrm{id} + (q - q^{-1})PR$ where $R_{21} = PRP$ has the tensor factors swapped.

It is already proven in [19, Proposition 10.5.1] that $A(R)$ in the $q$-Hecke case is an additive braided Hopf algebra in the braided category of $A(R)^{\mathrm{cop}}\otimes A(R)$-right comodules (or $A(R)$-bicomodules) with coproduct $\underline{\Delta}\mathbf{t} = \mathbf{t}\otimes 1 + 1\otimes\mathbf{t}$, or in a compact notation $\mathbf{t}'' = \mathbf{t}' + \mathbf{t}$ where $\mathbf{t}'$ is a second copy of $\mathbf{t}$, and $\mathbf{t}''$ obeys FRT bialgebra relation provided $\mathbf{t}'_1\mathbf{t}_2 = R_{21}\mathbf{t}_2\mathbf{t}'_1 R$. This expresses the braided Hopf algebra homomorphism property of the coproduct with respect to the relevant braiding $\Psi(\mathbf{t}_1\otimes\mathbf{t}_2) = R_{21}\mathbf{t}_2\otimes\mathbf{t}_1 R$, see [19] for details. As a consequence, as for any additive braided Hopf algebra, it followed that $A(R)$ has a bicovariant exterior algebra $\Omega(A(R))$ generated by $\mathbf{t}$ and $\mathrm{d}\mathbf{t}$ with bimodule relations as in the next lemma. The new part is that this makes $\Omega(A(R))$ strongly bicovariant.

**Lemma 6.3.1.** *Let $A(R)$ be the FRT-bialgebra with $R$ q-Hecke. The exterior algebra $\Omega(A(R))$ with bimodule and exterior algebra relations $(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 = R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R$ and $\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2 = -R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R$ as in [19] is strongly bicovariant with $\Delta_*\mathrm{d}\mathbf{t} = \mathrm{d}\mathbf{t}\otimes\mathbf{t} + \mathbf{t}\otimes\mathrm{d}\mathbf{t}$.*

*Proof.* The exterior algebra was already constructed in [19], but we provide a short check of the Leibniz rule so as to be self-contained. Thus, on generators,

$$\begin{aligned}
\mathrm{d}(R\mathrm{d}\mathbf{t}_1\mathbf{t}_2) = {}& R((\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 + \mathbf{t}_1\mathrm{d}\mathbf{t}_2) = RR_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + R\mathbf{t}_1\mathrm{d}\mathbf{t}_2\\
={}&\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + (q-q^{-1})RP\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + R\mathbf{t}_1\mathrm{d}\mathbf{t}_2 = R\mathbf{t}_1\mathrm{d}\mathbf{t}_2(\mathrm{id} + (q-q^{-1})PR) + \mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\\
={}&(\mathrm{d}\mathbf{t}_2)\mathbf{t}_1 R + \mathbf{t}_2\mathrm{d}\mathbf{t}_1 R = R\mathbf{t}_1\mathrm{d}\mathbf{t}_2 R_{21}R + \mathbf{t}_2\mathrm{d}\mathbf{t}_1 R = \mathrm{d}(\mathbf{t}_2\mathbf{t}_1 R)
\end{aligned}$$

where we used $R_{21} = R^{-1} + (q-q^{-1})P$ for the second equality and $\mathbf{t}_1\mathrm{d}\mathbf{t}_2 P = P\mathbf{t}_2\mathrm{d}\mathbf{t}_1$ for the third equality. Applying d once more to the stated bimodule relation on degree 1 gives the stated relations in degree 2 and there are no further relations in higher degree, which means that $\Omega(A(R))$ is the maximal prolongation of $\Omega^1(A(R))$.

The new part is the super-coproduct $\Delta_*$, which is uniquely determined by the super-coderivation property but we need to check that it is well-defined. Thus,

$$\begin{aligned}
\Delta_*((\mathrm{d}\mathbf{t}_1)\mathbf{t}_2) ={}&(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2\otimes\mathbf{t}_1\mathbf{t}_2 + \mathbf{t}_1\mathbf{t}_2\otimes(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 = R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\otimes\mathbf{t}_1\mathbf{t}_2 + \mathbf{t}_1\mathbf{t}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\\
={}&R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1 R + R_{21}\mathbf{t}_2\mathbf{t}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R = \Delta_*(R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R)
\end{aligned}$$

on $\Omega^1(A(R))$. Since $\Omega(A(R))$ is the maximal prolongation, we do not in principle need to check the relations in higher degrees due to arguments similar to the proof of Lemma 5.1.2. In practice, however, we check the degree 2 relations explicitly. Thus

$$\begin{aligned}
\Delta_*(-\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2) ={}& -\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes\mathbf{t}_1\mathbf{t}_2 - (\mathrm{d}\mathbf{t}_1)\mathbf{t}_2\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 - \mathbf{t}_1\mathbf{t}_2\otimes\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2\\
={}&R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\otimes\mathbf{t}_1\mathbf{t}_2 - R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + \mathbf{t}_1\mathbf{t}_2\otimes R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\\
={}&R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1 R - R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + R_{21}\mathbf{t}_2\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R
\end{aligned}$$

$$\begin{aligned}\Delta_*(R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R) = R_{21}\big(&\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1+\mathrm{d}\mathbf{t}_2.\mathbf{t}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1-\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2.\mathbf{t}_1+\mathbf{t}_2\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\big)R\\
=&R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1R+R_{21}R\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1R-R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}R\\
&+R_{21}\mathbf{t}_2\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R\\
=&R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1R+(\mathrm{id}+(q-q^{-1})PR)\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1R\\
&-R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2(\mathrm{id}+(q-q^{-1})P)+R_{21}\mathbf{t}_2\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R\\
=&R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathbf{t}_2\mathbf{t}_1R+\mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R+(q-q^{-1})PR\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1R\\
&-R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2-(q-q^{-1})R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2PR+R_{21}\mathbf{t}_2\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1R.\end{aligned}$$

The two expressions are equal since

$$\begin{aligned}PR\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1R=&R_{21}P\mathbf{t}_1\mathrm{d}\mathbf{t}_2\otimes\mathrm{d}\mathbf{t}_1.\mathbf{t}_2=R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes P\mathrm{d}\mathbf{t}_1.\mathbf{t}_2=R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes\mathrm{d}\mathbf{t}_2.\mathbf{t}_1P\\
=&R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}P=R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2PR.\end{aligned}$$

so that $(q-q^{-1})(PR\mathbf{t}_1\mathrm{d}\mathbf{t}_2R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1R-R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2PR)$ vanishes. □

Now consider additive braided Hopf algebras $V(R)$ on which $A(R)$ right coacts. These are the $q$-Hecke case of the general construction in [19, Proposition 10.2.8] and are generated by $\mathbf{x}=(x_i)$ regarded as a vector row with relations $q\mathbf{x}_1\mathbf{x}_2=\mathbf{x}_2\mathbf{x}_1R$ with coaction $\Delta_R\mathbf{x}=\mathbf{x}\otimes\mathbf{t}$. The braided Hopf algebra structure is expressed in [19, Theorem 10.2.6] as braided addition $\mathbf{x}''=\mathbf{x}'+\mathbf{x}$ where $\mathbf{x}''$ obeys the relation of $V(R)$ provided $\mathbf{x}'_1\mathbf{x}_2=\mathbf{x}_2\mathbf{x}'_1qR$ (the latter expresses the braiding $\Psi(\mathbf{x}_1\otimes\mathbf{x}_2)=\mathbf{x}_2\otimes\mathbf{x}_1qR$ induced by the coquasitriangular structure $\mathcal{R}(\mathbf{t}_1\otimes\mathbf{t}_2)=qR$). As before, the additive braided Hopf algebra theory implies the maximal prolongation exterior algebra $\Omega(V(R))$ given at the end of [19, Chapter 10.4] with the relations shown in the next lemma. A more formal treatment of the exterior algebras on an additive braided Hopf algebra, which underlies both $\Omega(A(R))$ and $\Omega(V(R))$, recently appeared in [35, Prop. 2.9]. The new part is that $\Delta_R$ is differentiable.

**Lemma 6.3.2.** *Let $V(R)$ be the right $A(R)$-covariant braided plane with $R$ $q$-Hecke. The exterior algebra $\Omega(V(R))$ with bimodule and exterior algebra relations* $(\mathrm{d}\mathbf{x}_1)\mathbf{x}_2=$

$\mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR$ *and* $-\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2 = \mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR$ *as in [19] has differentiable right coaction with* $\Delta_{R*}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}\otimes\mathbf{t} + \mathbf{x}\otimes\mathrm{d}\mathbf{t}$.

*Proof.* That $\Delta_{R*}$ is well-defined on degree 1 is

$$\begin{aligned}\Delta_{R*}((\mathrm{d}\mathbf{x}_1)\mathbf{x}_2) =&(\mathrm{d}\mathbf{x}_1)\mathbf{x}_2\otimes\mathbf{t}_1\mathbf{t}_2 + \mathbf{x}_1\mathbf{x}_2\otimes(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 = \mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR\otimes\mathbf{t}_1\mathbf{t}_2 + \mathbf{x}_1\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\\ =&\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes qR\mathbf{t}_1\mathbf{t}_2 + \mathbf{x}_1\mathbf{x}_2 R_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R = \mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 qR + \mathbf{x}_2\mathbf{x}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1 qR\\ =&\Delta_{R*}(\mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR).\end{aligned}$$

This is sufficient by Lemma 5.1.2 since $\Omega(V(R))$ is the maximal prolongation of $\Omega^1(V(R))$. If one wants to see it explicitly on degree 2, this is

$$\begin{aligned}\Delta_{R*}(-\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2) =& -\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes\mathbf{t}_1\mathbf{t}_2 - (\mathrm{d}\mathbf{x}_1)\mathbf{x}_2\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2 - \mathbf{x}_1\mathbf{x}_2\otimes\mathrm{d}\mathbf{t}_1\mathrm{d}\mathbf{t}_2\\ =&\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR\otimes\mathbf{t}_1\mathbf{t}_2 - q\mathbf{x}_2\mathrm{d}\mathbf{x}_1 R\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + \mathbf{x}_1\mathbf{x}_2\otimes R_{21}\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\\ =&\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 qR - q\mathbf{x}_2\mathrm{d}\mathbf{x}_1 R\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R + \mathbf{x}_2\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 qR.\end{aligned}$$

$$\begin{aligned}\Delta_{R*}(\mathbf{x}_2\mathrm{d}\mathbf{x}_1 qR) =& (\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 + \mathrm{d}\mathbf{x}_2.\mathbf{x}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1 - \mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes(\mathrm{d}\mathbf{t}_2)\mathbf{t}_1 + \mathbf{x}_2\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1)qR\\ =&\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 qR + q^2\mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R - q\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2 R_{21}R + \mathbf{x}_2\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 qR\\ =&\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 qR + q^2\mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R - q\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2(\mathrm{id} + (q-q^{-1})PR)\\ &+ \mathbf{x}_2\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 qR\\ =&\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 qR + q^2\mathbf{x}_1\mathrm{d}\mathbf{x}_2\otimes R_{21}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R - q\mathbf{x}_2\mathrm{d}\mathbf{x}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2 + \mathbf{x}_2\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{t}_1 qR\\ &- (q^2-1)\mathbf{x}_2\mathrm{d}\mathbf{x}_1 PR_{21}\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1 R\end{aligned}$$

which simplifies to the first expression. □

Now suppose $A(R)$ has a central grouplike element $D$ such that $A = A(R)[D^{-1}]$ is a Hopf algebra. For the standard $\mathbb{C}_q[GL_n]$ $R$-matrix, this is just the $q$-determinant allowing $S\mathbf{t}$ to be constructed as $D^{-1}$ times a $q$-matrix of cofactors. (Another approach, which we will not take, is to assume that $R$ is 'bi-invertible' and define a Hopf algebra by reconstruction from a rigid braided category defined by $R$.) It is easy to see that $\Omega(A(R))$ extends to a strongly bicovariant exterior algebra $\Omega(A)$ with $\mathrm{d}D^{-1} = -D^{-1}(\mathrm{d}D)D^{-1}$. We take the same $\Delta_R\mathbf{x} = \mathbf{x}\otimes\mathbf{t}$ on $V(R)$ but now viewed as a coaction of $A$. We are now ready for our main theorem.

**Theorem 6.3.3.** *Let $A = A(R)[D^{-1}]$ with $R$ $q$-Hecke and $V(R)$ the right-covariant braided plane as above. Then $\Omega(V(R))$ is a super-braided Hopf algebra with $x_i, \mathrm{d}x_i$ primitive in the crossed module category $\mathcal{M}^{\Omega(A)}_{\Omega(A)}$ with coaction $\Delta_{R*}$ as in Lemma 6.3.2 and*

$$\mathbf{x}_1{\triangleleft}\mathbf{t}_2 = \mathbf{x}_1 q^{-1}R_{21}^{-1},\quad (\mathrm{d}\mathbf{x}_1){\triangleleft}\mathbf{t}_2 = \mathrm{d}\mathbf{x}_1 q^{-1}R,\quad \mathbf{x}_1{\triangleleft}\mathrm{d}\mathbf{t}_2 = (q^{-2}-1)\mathrm{d}\mathbf{x}_1 P,\quad (\mathrm{d}\mathbf{x}_1){\triangleleft}\mathrm{d}\mathbf{t}_2 = 0,$$

*where $P$ is a permutation matrix. This defines a strongly bicovariant exterior algebra $\Omega(A{\triangleright\!\!<}V(R)) := \Omega(A){\triangleright\!\!<}\Omega(V(R))$ with relations, coproducts, and antipodes*

$$\mathbf{x}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{x}_1 q^{-1}R_{21}^{-1},\quad (\mathrm{d}\mathbf{x}_1)\mathbf{t}_2 = \mathbf{t}_2\mathrm{d}\mathbf{x}_1 q^{-1}R,\quad \mathbf{x}_1\mathrm{d}\mathbf{t}_2 = (\mathrm{d}\mathbf{t}_2)\mathbf{x}_1 q^{-1}R_{21}^{-1} + (q^{-2}-1)\mathbf{t}_2\mathrm{d}\mathbf{x}_1 P$$

$$\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{t}_2 = -\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{x}_1 q^{-1}R,\quad \Delta\mathbf{x} = 1\otimes\mathbf{x} + \mathbf{x}\otimes\mathbf{t},\quad \Delta_*\mathrm{d}\mathbf{x} = 1\otimes\mathrm{d}\mathbf{x} + \mathrm{d}\mathbf{x}\otimes\mathbf{t} + \mathbf{x}\otimes\mathrm{d}\mathbf{t}$$

$$S\mathbf{x} = -\mathbf{x}S\mathbf{t},\quad S\mathrm{d}\mathbf{x} = -(\mathrm{d}\mathbf{x})S\mathbf{t} - \mathbf{x}S\mathrm{d}\mathbf{t}.$$

*Proof.* Note that the degree 0 part is the Hopf algebra $A{\triangleright\!\!<}V(R)$ and is just the $q$-Hecke case of construction of inhomogeneous quantum groups by cobosonisation in [19]. Note that the action of $\Omega^1(A)$ is not given as far as we know by some general construction and one has to check by hand that we indeed obtain $\Omega(V(R))$ as a super right $\Omega(A)$-crossed module as follow. First, we have

$$\begin{aligned}\Delta_R(\mathbf{x}_1{\triangleleft}\mathbf{t}_2) =& \mathbf{x}_1{\triangleleft}\mathbf{t}_2\otimes(S\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2 = \mathbf{x}_1 q^{1}R_{21}^{-1}\otimes(S\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2 = (\mathrm{id}\otimes S\mathbf{t}_2)(\mathbf{x}_1 q^{1}R_{21}^{-1}\otimes\mathbf{t}_1\mathbf{t}_2)\\ =&(\mathrm{id}\otimes S\mathbf{t}_2)(\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1 q^{-1}R_{21}^{-1}) = \mathbf{x}_1\otimes(S\mathbf{t}_2)\mathbf{t}_2\mathbf{t}_1 q^{-1}R_{21}^{-1}) = \mathbf{x}_1\otimes\mathbf{t}_1 q^{-1}R_{21}^{-1}\end{aligned}$$

Next, we have

$$\begin{aligned}\Delta_{R*}((\mathrm{d}\mathbf{x}_1){\triangleleft}\mathbf{t}_2) =&(\mathrm{d}\mathbf{x}_1){\triangleleft}\mathbf{t}_2\otimes(S\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2+\mathbf{x}_1{\triangleleft}\mathbf{t}_2\otimes(S\mathbf{t}_2)(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2\\
=&\mathrm{d}\mathbf{x}_1q^{-1}R\otimes(S\mathbf{t}_2)\mathbf{t}_1\mathbf{t}_2+\mathbf{x}_1q^{-1}R_{21}^{-1}\otimes(S\mathbf{t}_2)(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2\\
=&(\mathrm{id}\otimes S\mathbf{t}_2)(\mathrm{d}\mathbf{x}_1q^{-1}R\otimes\mathbf{t}_1\mathbf{t}_2+\mathbf{x}_1q^{-1}R_{21}^{-1}\otimes(\mathrm{d}\mathbf{t}_1)\mathbf{t}_2)\\
=&(\mathrm{id}\otimes S\mathbf{t}_2)(\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1q^{-1}R+\mathbf{x}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1q^{-1}R)\\
=&\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_1q^{-1}R+\mathbf{x}_1\otimes\mathrm{d}\mathbf{t}_1q^{-1}R.\end{aligned}$$

This verifies the conditions of Lemma 6.2.2 (i) hold and therefore $\Omega(V(R))$ is a super right $\Omega(A)$-crossed module, but one can also check directly that $\Delta_{R*}(\mathbf{x}_1{\triangleleft}\mathrm{d}\mathbf{t}_2)$ and $\Delta_{R*}((\mathrm{d}\mathbf{x}_1){\triangleleft}\mathrm{d}\mathbf{t}_2)$ obey the crossed module axiom.

One can also check that $\Omega(V(R))$ is indeed a super-Hopf algebra in this category with $x_i, \mathrm{d}x_i$ primitive with the braiding for $\Omega(V(R))$ in the crossed module category comes out from the action and coaction of $\Omega(A)$ as

$$\Psi(\mathbf{x}_1\otimes\mathbf{x}_2)=\mathbf{x}_2\otimes\mathbf{x}_1q^{-1}R_{21}^{-1},\quad \Psi(\mathrm{d}\mathbf{x}_1\otimes\mathbf{x}_2)=\mathbf{x}_2\otimes\mathrm{d}\mathbf{x}_1q^{-1}R$$

$$\Psi(\mathbf{x}_1\otimes\mathrm{d}\mathbf{x}_2)=\mathrm{d}\mathbf{x}_2\otimes\mathbf{x}_1q^{-1}R_{21}^{-1}+(q^{-2}-1)\mathbf{x}_2\otimes\mathbf{x}_1P,\quad \Psi(\mathrm{d}\mathbf{x}_1\otimes\mathrm{d}\mathbf{x}_2)=\mathrm{d}\mathbf{x}_2\otimes\mathbf{x}_1q^{-1}R,$$

and $V(R)$ is a braided Hopf algebra with respect to this action. This verifies that the conditions of Lemma 6.2.2 (ii) hold. We exhibit directly that the construction works by showing that the super coproduct $\Delta_*$ of $\Omega(A{\triangleright\!\!\!<}V(R))$ is indeed well-defined degree 1. We let $\lambda=q^{-2}-1$. Then,

$$\begin{aligned}\Delta_*(\mathbf{x}_1\mathrm{d}\mathbf{t}_2) =&\mathrm{d}\mathbf{t}_2\otimes\mathbf{x}_1\mathbf{t}_2+\mathbf{t}_2\otimes\mathbf{x}_1\mathrm{d}\mathbf{t}_2+\mathbf{x}_1\mathrm{d}\mathbf{t}_2\otimes\mathbf{t}_1\mathbf{t}_2+\mathbf{x}_1\mathbf{t}_2\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2\\
=&\mathrm{d}\mathbf{t}_2\otimes\mathbf{t}_2\mathbf{x}_1q^{-1}R_{21}^{-1}+\mathbf{t}_2\otimes\mathrm{d}\mathbf{t}_2.\mathbf{x}_1q^{-1}R_{21}^{-1}+\lambda\mathbf{t}_2\otimes\mathbf{t}_2\mathrm{d}\mathbf{x}_1P\\
&+\mathrm{d}\mathbf{t}_2\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1q^{-1}R_{21}^{-1}+\lambda\mathbf{t}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1P+\mathbf{t}_2\mathbf{x}_1q^{-1}R_{21}^{-1}\otimes\mathbf{t}_1\mathrm{d}\mathbf{t}_2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Delta_*((\mathrm{d}\mathbf{t}_2)\mathbf{x}_1 q^{-1}R_{21}^{-1}) &= (\mathrm{d}\mathbf{t}_2\otimes\mathbf{t}_2\mathbf{x}_1+\mathrm{d}\mathbf{t}_2.\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1+\mathbf{t}_2\otimes(\mathrm{d}\mathbf{t}_2)\mathbf{x}_1+\mathbf{t}_2\mathbf{x}_1\otimes(\mathrm{d}\mathbf{t}_2)\mathbf{t}_1)q^{-1}R_{21}^{-1}\\ &=\Big(\mathrm{d}\mathbf{t}_2\otimes\mathbf{t}_2\mathbf{x}_1+\mathrm{d}\mathbf{t}_2.\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1+\mathbf{t}_2\otimes(\mathrm{d}\mathbf{t}_2)\mathbf{x}_1\Big)q^{-1}R_{21}^{-1}+q^{-1}\mathbf{t}_2\mathbf{x}_1\otimes R\mathbf{t}_1\mathrm{d}\mathbf{t}_2\end{aligned}$$

$$\Delta_*(\lambda\mathbf{t}_2\mathrm{d}\mathbf{x}_1P) =(\mathbf{t}_2\otimes\mathbf{t}_2\mathrm{d}\mathbf{x}_1+\mathbf{t}_2\mathrm{d}\mathbf{x}_1\otimes\mathbf{t}_2\mathbf{t}_1+\mathbf{t}_2\mathbf{x}_1\otimes\mathbf{t}_2\mathrm{d}\mathbf{t}_1)\lambda P$$

from which we find that $\Delta_*(\mathbf{x}_1\mathrm{d}\mathbf{t}_2-\mathrm{d}\mathbf{t}_2.\mathbf{x}_1q^{-1}R_{21}^{-1}-\lambda\mathbf{t}_2\mathrm{d}\mathbf{x}_1P)=0$ using $\lambda P=q^{-1}(R_{21}^{-1}-R)$. We also exhibit that this bimodule relation is compatible with the graded-Leibniz rule as it must,

$$\begin{aligned}&\mathrm{d}(\mathbf{x}_1\mathrm{d}\mathbf{t}_2-(\mathrm{d}\mathbf{t}_2)\mathbf{x}_1q^{-1}R_{21}^{-1}-\lambda\mathbf{t}_2\mathrm{d}\mathbf{x}_1P)\\ &\qquad=\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{t}_2+\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{x}_1q^{-1}R_{21}^{-1}-\lambda\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{x}_1P=\mathrm{d}\mathbf{t}_2\mathrm{d}\mathbf{x}_1\Big(q^{-1}(-R+R_{21}^{-1})-\lambda P\Big)=0.\end{aligned}$$

Similarly for the other relations. □

We also know by Corollary 6.2.3 that the right $A{\triangleright\!\!\!<}V(R)$ coaction $\Delta_R\mathbf{x}=\mathbf{x}\otimes\mathbf{t}+1\otimes\mathbf{x}\in V(R)\otimes A{\triangleright\!\!\!<}V(R)$ on $V(R)$ coming view of $A{\triangleright\!\!\!<}V(R)$ as inhomogeneous quantum groups, is differentiable for the exterior algebras above.

## 6.4 Calculations for the smallest $R$-matrices

The construction in Theorem 6.3.3 includes the standard $q$-deformation $R$-matrix for the $SL_n$ for all $n$ as these are all known to be $q$-Hecke when normalised correctly, see [19]. In this case $A=\mathbb{C}_q[GL_n]$ and $V(R)=\mathbb{C}_q^n$ is the standard quantum-braided plane with relations $x_jx_i=qx_ix_j$ for all $j>i$. Thus we obtain $\Omega(\mathbb{C}_q[GL_n]{\triangleright\!\!\!<}\mathbb{C}_q^n)$ such that the canonical coaction of $\mathbb{C}_q[GL_n]{\triangleright\!\!\!<}\mathbb{C}_q^n$ on $\mathbb{C}_q^n$ is differentiable. In this section, we show $n=1$ and $n=2$ explicitly.

For $n=1$, $R=(q)$, $D=t$ and $A=\mathbb{C}[t,t^{-1}]$ is the algebraic circle with $\Delta t=t\otimes t$ which has a strongly bicovariant exterior algebra with

$$(\mathrm{d}t)t = q^2 t\mathrm{d}t, \quad (\mathrm{d}t)^2 = 0, \quad \Delta_* \mathrm{d}t = t \otimes \mathrm{d}t + \mathrm{d}t \otimes t.$$

by Lemma 6.3.1 and those implied for $t^{-1}$. (This is the standard bicovariant calculus a circle, for some parameter $q$). We have $B = V(R) = \mathbb{C}[x]$ with calculus $(\mathrm{d}x)x = q^2 x\mathrm{d}x$, $(\mathrm{d}x)^2 = 0$, which is $A$-covariant with $\Delta_R x = x \otimes t$ by Lemma 6.3.2.

**Proposition 6.4.1.** *We can view $B = V(R) = \mathbb{C}[x]$ as a braided Hopf algebra in $\mathcal{M}_A^A$ as part of $\Omega(B)$ a super braided Hopf algebra in $\mathcal{M}_{\Omega(A)}^{\Omega(A)}$ with*

$$x{\triangleleft}t = q^{-2}x, \quad x{\triangleleft}\mathrm{d}t = (q^{-2} - 1)\mathrm{d}x, \quad \mathrm{d}x{\triangleleft}t = \mathrm{d}x, \quad \mathrm{d}x{\triangleleft}\mathrm{d}t = 0$$

$$\Delta_R x = x \otimes t, \quad \Delta_{R*}\mathrm{d}x = \mathrm{d}x \otimes t + x \otimes \mathrm{d}t.$$

$$(\mathrm{d}x)x = q^2 x\mathrm{d}x, \quad (\mathrm{d}x)^2 = 0, \quad \underline{\Delta} x = 1 \otimes x + x \otimes 1, \quad \underline{\Delta}_* \mathrm{d}x = 1 \otimes \mathrm{d}x + \mathrm{d}x \otimes 1.$$

*The super bosonisation defines a strongly bicovariant exterior algebra $\Omega(\mathbb{C}_q[B_+]) := \Omega(A){\triangleright\!\!\!<}\Omega(B)$ with relations and comultiplication*

$$xt = q^{-2}tx, \quad (\mathrm{d}x)t = t\mathrm{d}x, \quad (\mathrm{d}t)x = q^2 x\mathrm{d}t + (q^2 - 1)t\mathrm{d}x, \quad \mathrm{d}x\mathrm{d}t = -\mathrm{d}t\mathrm{d}x$$

$$\Delta x = 1 \otimes x + x \otimes t, \quad \Delta_* \mathrm{d}x = 1 \otimes \mathrm{d}x + \mathrm{d}x \otimes t + x \otimes \mathrm{d}t$$

*where we identify $A{\triangleright\!\!\!<}B = \mathbb{C}_q[B_+]$ the quantisation of the positive Borel subgroup of $SL_2$ (a quotient of $\mathbb{C}_q[SL_2]$). Moreover, the coaction $\Delta_R : \mathbb{C}[x] \to \mathbb{C}[x] \otimes \mathbb{C}[B_+]$ given by $\Delta_R x = 1 \otimes x + x \otimes t$ is differentiable.*

*Proof.* This is read off immediately from Theorem 6.3.3 but it is simple enough to verify the key facts by hand. $\square$

**Remark 6.4.2.** The Hopf algebra $\mathbb{C}_q[B_+]$ is also called the Sweedler-Taft algebra (but we think of it as a $q$-deformed coordinate algebra). One can also think of it as $U_q(b_+)$ and in this case the exterior algebra is the two dimensional $\Omega(U_q(b_+))$ found in [41]. In addition, we can work on $q$ primitive $n$-th odd root of unity case where now $A = \mathbb{C}_q[t]/(t^n - 1)$ and $B = \mathbb{C}[x]/(x^n)$, giving us the the exterior algebra of the reduced quantum group $c_q[B_+]$ with additional relations $t^n = 1$ and $x^n = 0$.

We now compute the rather more complicated $n = 2$ case where $A = \mathbb{C}_q[GL_2]$. This is explained in Section 4.3 but now we write its generators as $t^1{}_1 = a$, $t^1{}_2 = b$, etc. It is known for example in [7, 47] that $\mathbb{C}_q[GL_2]$ has an obvious 1-parameter family of coquasitriangular structures

$$\mathcal{R}(\mathbf{t}_1 \otimes \mathbf{t}_2) = R_\alpha = q^\alpha \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{6.4.1}$$

with $R = R_0$ the $q$-Hecke normalisation. The choice of $\mathcal{R}$ means a 1-parameter family $\Omega_\alpha(\mathbb{C}_q[GL_2])$ it we use the standard construction for coquasitriangular Hopf algebras. We refer to [35] for a recent treatment, which gives this calculus in the form $\mathbb{C}_q[GL_2] \triangleright\!\!< \Lambda$ where the left-invariant 1-forms $\Lambda^1$ has basis $E_1{}^1 = e_a, E_1{}^2 = e_b, E_2{}^1 = e_c$ and $E_2{}^2 = e_d$ of $\Lambda^1$ and is a right crossed $A$-module by

$$\Delta_R E_r{}^s = E_m{}^n \otimes t^m{}_r S t^s{}_n, \quad E_r{}^s \triangleleft t^i{}_j = E_m{}^n (R_\alpha)^m{}_r{}^i{}_k (R_\alpha)^k{}_j{}^s{}_n$$

resulting in super coproduct and relations

$$\Delta_* E_r{}^s = 1 \otimes E_r{}^s + E_m{}^n \otimes t^m{}_r S t^s{}_n, \quad E_r{}^s t^i{}_j = t^i{}_k E_m{}^n (R_\alpha)^m{}_r{}^k{}_l (R_\alpha)^l{}_j{}^s{}_n.$$

Explicitly, this is

$$e_a \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{2\alpha} \begin{pmatrix} q^2 a & b \\ q^2 c & d \end{pmatrix} e_a, \quad [e_b, \begin{pmatrix} a & b \\ c & d \end{pmatrix}]_{q^{1+2\alpha}} = q^{1+2\alpha} \lambda \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} e_a$$

$$[e_c, \begin{pmatrix} a & b \\ c & d \end{pmatrix}]_{q^{1+2\alpha}} = q^{1+2\alpha}\lambda \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} e_a, \quad [e_d, \begin{pmatrix} a \\ c \end{pmatrix}]_{q^{2\alpha}} = q^{2\alpha}\lambda \begin{pmatrix} b \\ d \end{pmatrix} e_b$$

$$[e_d, \begin{pmatrix} b \\ d \end{pmatrix}]_{q^{2+2\alpha}} = q^{2\alpha}\lambda \begin{pmatrix} ae_c + \lambda be_a \\ ce_c + \lambda de_a \end{pmatrix}$$

where $\lambda = q - q^{-1}$. When $\alpha = -\frac{1}{2}$ the calculus descends to the quotient $D = 1$ giving the standard 4D calculus on $\mathbb{C}_q[SL_2]$ as in [47] but otherwise we are in the same family but with a different $q$-factor in the commutation relations. It is inner with $\theta = e_a + e_d$, so that

$$\mathrm{d}t^i{}_j = t^i{}_k(R_{\alpha 21}R_\alpha)^k{}_j{}^r{}_s E_r{}^s - t^i{}_j\theta = [\theta, t^i{}_j].$$

On the other hand, the crossed module braiding $\Psi$ on $\Lambda^1$ is given in [35] is given by an expression with an equal number of $R_\alpha$ and its appropriate inverse, so does not depend on the normalisation factor in $R_\alpha$ and hence the left-invariant exterior algebra $\Lambda = B_-(\Lambda^1) = T\Lambda^1/\ker(\mathrm{id} - \Psi)$ is the same as for the standard 4D calculus on $\mathbb{C}_q[SL_2]$, namely the usual Grassmanian variables on $e_a, e_b, e_c$ and

$$e_ae_d + e_de_a + q^{-1}\lambda e_ce_b = 0, \quad e_de_c + q^2e_ce_d + q^{-1}\lambda e_ae_c = 0$$

$$e_be_d + q^2e_de_b + q^{-1}\lambda e_be_a = 0, \quad e_a^2 = e_b^2 = e_c^2 = 0, \quad e_d^2 = q^{-1}\lambda e_ce_b$$

and with exterior derivative

$$\mathrm{d}e_a = q^{-1}\lambda e_be_c, \quad \mathrm{d}e_b = \lambda(q^{-1}e_a - qe_d)e_b, \quad \mathrm{d}e_c = \lambda e_c(q^{-1}e_a - qe_d), \quad \mathrm{d}e_d = q^{-1}\lambda e_ce_b.$$

This gives a full structure on exterior algebra $\Omega_\alpha(\mathbb{C}_q[GL_2]) = \mathbb{C}_q[GL_2]{\triangleright\!\!\!<}\Lambda$.

We next consider the quantum-braided plane $B = \mathbb{C}_q^2$ generated by $x_1, x_2$ with relation $x_2x_1 = qx_1x_2$ and viewed initially as a braided Hopf algebra in the category of right $\mathbb{C}_q[GL_2]$ modules and with exterior algebra $\Omega(\mathbb{C}_q^2)$ with standard relations and coaction from Lemma 6.3.2,

$$(\mathrm{d}x_i)x_i = q^2 x_i \mathrm{d}x_i, \quad (\mathrm{d}x_1)x_2 = qx_2\mathrm{d}x_1, \quad (\mathrm{d}x_2)x_1 = qx_1\mathrm{d}x_2 + (q^2-1)x_2\mathrm{d}x_1$$
$$(\mathrm{d}x_i)^2 = 0, \quad \mathrm{d}x_2\mathrm{d}x_1 = -q^{-1}\mathrm{d}x_1\mathrm{d}x_2, \quad \Delta_R x_i = x_j \otimes t^j{}_i. \tag{6.4.2}$$

**Lemma 6.4.3.** *The 4D strongly bicovariant calculus $\Omega(\mathbb{C}_q[GL_2])$ in Lemma 6.3.1 has relations*

$$(\mathrm{d}a)a = q^2 a\mathrm{d}a, \quad (\mathrm{d}a)b = qb\mathrm{d}a, \quad (\mathrm{d}a)c = qc\mathrm{d}a, \quad (\mathrm{d}a)d = d\mathrm{d}a$$

$$(\mathrm{d}b)a = qa\mathrm{d}b + (q^2-1)b\mathrm{d}a, \quad (\mathrm{d}b)b = q^2 b\mathrm{d}b, \quad (\mathrm{d}b)c = c\mathrm{d}b + (q-q^{-1})d\mathrm{d}a, \quad (\mathrm{d}b)d = qd\mathrm{d}b$$

$$(\mathrm{d}c)a = qa\mathrm{d}c + (q^2-1)c\mathrm{d}a, \quad (\mathrm{d}c)b = b\mathrm{d}c + (q-q^{-1})d\mathrm{d}a, \quad (\mathrm{d}c)c = q^2 c\mathrm{d}c, \quad (\mathrm{d}c)d = qd\mathrm{d}c$$

$$(\mathrm{d}d)a = a\mathrm{d}d + (q-q^{-1})(b\mathrm{d}c + c\mathrm{d}b + (q-q^{-1})d\mathrm{d}a), \quad (\mathrm{d}d)b = qb\mathrm{d}d + (q^2-1)d\mathrm{d}b$$

$$(\mathrm{d}d)c = qc\mathrm{d}d + (q^2-1)d\mathrm{d}c, \quad (\mathrm{d}d)d = q^2 d\mathrm{d}d$$

*along with implied relations for $D^{-1}$, and is isomorphic to $\Omega_0(\mathbb{C}_q[GL_2])$. Moreover, it is the unique strongly bicovariant exterior algebra containing strongly bicovariant $\Omega(\mathbb{C}[D, D^{-1}])$ such that the canonical right coaction on $\mathbb{C}_q[GL_2]$ on $\mathbb{C}_q^2$ is differentiable.*

*Proof.* The displayed calculus is a routine calculation from the Lemma 6.3.1, while the last part of the statement is as follows. Requiring $\Delta_{R*}((\mathrm{d}x_i)x_i) = \Delta_{R*}(q^2 x_i \mathrm{d}x_i)$ we obtain all the relations stated except for those stated for $(\mathrm{d}a)d, (\mathrm{d}b)c$, $(\mathrm{d}c)b$, $(\mathrm{d}d)a$, $(\mathrm{d}c)\mathrm{d}b$ and $(\mathrm{d}d)\mathrm{d}a$, but we also get the following additional conditions

$$(\mathrm{d}c)b + q^{-1}(\mathrm{d}a)d = b\mathrm{d}c + qd\mathrm{d}a, \quad (\mathrm{d}b)c + q^{-1}(\mathrm{d}a)d = c\mathrm{d}b + qd\mathrm{d}a$$

$$(\mathrm{d}c)b + q(\mathrm{d}d)a = q^2 b\mathrm{d}c + (q^2-1)c\mathrm{d}b + qa\mathrm{d}d + q(q^2-1)d\mathrm{d}a$$

$$(\mathrm{d}b)c + q(\mathrm{d}d)a = q^2 c\mathrm{d}b + (q^2-1)b\mathrm{d}c + qa\mathrm{d}d + q(q^2-1)d\mathrm{d}a.$$

Since $D$ is central, group-like and invertible, then $\mathbb{C}[D, D^{-1}]$ is a sub-Hopf algebra of $\mathbb{C}_q[GL_2]$ and is isomorphic to $\mathbb{C}[t, t^{-1}]$. By requiring $(\mathrm{d}D)D = q^2 D(\mathrm{d}D)$ as in Proposition 6.4.1, the above conditions are simplified into

$$a[\mathrm{d}a, d]d + qbc[\mathrm{d}a, d] = 0.$$

This can be simplified further by moving the elements of degree 0 to the right by using the already-known relations and obtain $[\mathrm{d}a, d]D = 0$, which implies $[\mathrm{d}a, d] = 0$ since $D \neq 0$. The rest of relations are followed. We are now able to write $\mathrm{d}D$ explicitly as

$$\mathrm{d}D = a\mathrm{d}d - q^{-1}b\mathrm{d}c - q^{-1}c\mathrm{d}b + q^{-2}d\mathrm{d}a$$

$$(\mathrm{d}t^i{}_j)D = q^2 D\mathrm{d}t^i{}_j, \quad (\mathrm{d}D)t^i{}_j = t^i{}_j\mathrm{d}D + (q^2 - 1)D\mathrm{d}t^i{}_j.$$

By applying d to the stated bimodule relations, we obtain

$$(\mathrm{d}a)^2 = (\mathrm{d}b)^2 = (\mathrm{d}c)^2 = (\mathrm{d}d)^2 = (\mathrm{d}D)^2 = 0$$

$$\mathrm{d}b\mathrm{d}a = -q^{-1}\mathrm{d}a\mathrm{d}b, \quad \mathrm{d}c\mathrm{d}a = -q^{-1}\mathrm{d}a\mathrm{d}c, \quad \mathrm{d}d\mathrm{d}b = -q^{-1}\mathrm{d}b\mathrm{d}c, \quad \mathrm{d}d\mathrm{d}c = -q^{-1}\mathrm{d}c\mathrm{d}d$$

$$\mathrm{d}c\mathrm{d}b = -\mathrm{d}b\mathrm{d}c + (q - q^{-1})\mathrm{d}a\mathrm{d}d, \quad \mathrm{d}d\mathrm{d}a = -\mathrm{d}a\mathrm{d}d, \quad \mathrm{d}D\mathrm{d}t^i{}_j = -q^{-2}\mathrm{d}t^i{}_j\mathrm{d}D$$

for the degree 2 relations.

Finally, working in this calculus, consider the basis of left-invariant 1-form $\omega_a = \varpi(a)$, $\omega_b = \varpi(b)$, $\omega_c = \varpi(c)$, $\omega_d = \varpi(d)$, where $\varpi(a) = (Sa_{(1)})\mathrm{d}a_{(2)}$ is the quantum Maurer-Cartan form. Explicitly we have basic forms and relations

$$\omega_a = D^{-1}(d\mathrm{d}a - qb\mathrm{d}c), \quad \omega_b = D^{-1}(d\mathrm{d}b - qb\mathrm{d}d)$$

$$\omega_c = D^{-1}(a\mathrm{d}c - q^{-1}c\mathrm{d}a), \quad \omega_d = D^{-1}(a\mathrm{d}d - q^{-1}c\mathrm{d}b)$$

$$\omega_a \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^2 a & b \\ q^2 c & d \end{pmatrix} \omega_a, \quad [\omega_b, \begin{pmatrix} a & b \\ c & d \end{pmatrix}]_q = \lambda \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \omega_a, \quad [\omega_b, \begin{pmatrix} a & b \\ c & d \end{pmatrix}]_q = \lambda \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \omega_a$$

$$[\omega_d, \begin{pmatrix} a \\ c \end{pmatrix}] = \begin{pmatrix} q\lambda b\omega_c + \lambda^2 a\omega_a \\ q\lambda d\omega_c + \lambda^2 c\omega_a \end{pmatrix}, \quad [\omega_d, \begin{pmatrix} b \\ d \end{pmatrix}]_{q^2} = \lambda \begin{pmatrix} a \\ c \end{pmatrix} \omega_b$$

where $\lambda = q - q^{-1}$, and exterior derivative

$$\mathrm{d}a = a\omega_a + b\omega_c, \quad \mathrm{d}b = a\omega_b + b\omega_d, \quad \mathrm{d}c = c\omega_a + d\omega_c, \quad c\omega_b + d\omega_d.$$

It is then a straightforward calculation to prove that $\varphi : \Omega(\mathbb{C}_q[GL_2]) \to \Omega_0(\mathbb{C}_q[GL_2])$ given by the identity map for elements of degree 0 and

$$\varphi(\omega_a) = q\lambda e_a, \quad \varphi(\omega_b) = \lambda e_c, \quad \varphi(\omega_c) = \lambda e_b, \quad \varphi(\omega_d) = q\lambda e_d + \lambda^2 e_a$$

is an isomorphism. □

We are now ready to state our example of Theorem 6.3.3.

**Proposition 6.4.4.** *Let $\Omega(A) = \Omega(\mathbb{C}_q[GL_2])$ be the strongly bicovariant exterior algebra in Lemma 6.4.3 and $B = \mathbb{C}_q^2$ be viewed as a braided Hopf algebra in $\mathcal{M}_A^A$ as part of $\Omega(B) = \Omega(\mathbb{C}_q^2)$ a super braided Hopf algebra in the category of $\Omega(\mathbb{C}_q[GL_2])$-crossed modules with (co)action and coproduct*

$$x_1 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-2}x_1 & (q^{-2}-1)x_2 \\ 0 & q^{-1}x_1 \end{pmatrix}, \quad x_2 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}x_2 & 0 \\ 0 & q^{-2}x_2 \end{pmatrix}$$

$$x_1 \triangleleft \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} (q^{-2}-1)\mathrm{d}x_1 & (q^{-2}-1)\mathrm{d}x_2 \\ 0 & 0 \end{pmatrix} \quad \mathrm{d}x_1 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{d}x_1 & 0 \\ 0 & q^{-1}\mathrm{d}x_1 \end{pmatrix}$$

$$x_2 \triangleleft \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (q^{-2}-1)\mathrm{d}x_1 & (q^{-2}-1)\mathrm{d}x_2 \end{pmatrix} \quad \mathrm{d}x_2 \triangleleft \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}\mathrm{d}x_2 & 0 \\ (1-q^{-2})\mathrm{d}x_1 & \mathrm{d}x_2 \end{pmatrix}$$

$$\mathrm{d}x_i{\triangleleft}\mathrm{d}t^k{}_l = 0, \quad \Delta_R x_1 = x_1 \otimes a + x_2 \otimes c, \quad \Delta_R x_2 = x_1 \otimes b + x_2 \otimes c$$

$$\Delta_{R*}\mathrm{d}x_1 = \mathrm{d}x_1 \otimes a + \mathrm{d}x_2 \otimes c + x_1 \otimes \mathrm{d}a + x_2 \otimes \mathrm{d}c$$

$$\Delta_{R*}\mathrm{d}x_2 = \mathrm{d}x_1 \otimes b + \mathrm{d}x_2 \otimes d + x_1 \otimes \mathrm{d}b + x_2 \otimes \mathrm{d}d$$

$$\underline{\Delta} x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \underline{\Delta}_* \mathrm{d}x_i = \mathrm{d}x_i \otimes 1 + 1 \otimes \mathrm{d}x_i.$$

*Its super bosonisation is a strongly bicovariant exterior algebra* $\Omega(\mathbb{C}_q[GL_2]{\ltimes\!\!\!\cdot}\mathbb{C}^2) := \Omega(\mathbb{C}_q[GL_2]){\ltimes\!\!\!\cdot}\Omega(\mathbb{C}_q^2)$ *with sub-exterior algebras* $\Omega(\mathbb{C}_q[GL_2])$, $\Omega(\mathbb{C}_q^2)$ *and cross relations and super coproduct*

$$x_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-2}ax_1 & q^{-1}bx_1 + (q^{-2}-1)ax_2 \\ q^{-2}cx_1 & q^{-1}dx_1 + (q^{-2}-1)cx_2 \end{pmatrix}, \quad x_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}ax_1 & q^{-2}bx_2 \\ q^{-1}cx_2 & q^{-2}dx_2 \end{pmatrix}$$

$$\mathrm{d}x_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\mathrm{d}x_1 & q^{-1}b\mathrm{d}x_1 \\ c\mathrm{d}x_1 & q^{-1}d\mathrm{d}x_1 \end{pmatrix}, \quad \mathrm{d}x_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}a\mathrm{d}x_2 + (1-q^{-2})b\mathrm{d}x_1 & b\mathrm{d}x_2 \\ q^{-1}c\mathrm{d}x_2 + (1-q^{-2})c\mathrm{d}x_2 & d\mathrm{d}x_2 \end{pmatrix}$$

$$x_1 \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} q^{-2}(\mathrm{d}a)x_1 + (q^{-2}-1)a\mathrm{d}x_1 & q^{-1}(\mathrm{d}b)x_1 + (q^{-2}-1)((\mathrm{d}a)x_2 + a\mathrm{d}x_2) \\ q^{-2}(\mathrm{d}c)x_1 + (q^{-2}-1)c\mathrm{d}x_1 & q^{-1}(\mathrm{d}d)x_1 + (q^{-2}-1)((\mathrm{d}c)x_2 + c\mathrm{d}x_2) \end{pmatrix}$$

$$x_2 \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} q^{-1}(\mathrm{d}a)x_2 + (q^{-2}-1)b\mathrm{d}x_1 & q^{-2}(\mathrm{d}b)x_2 + (q^{-2}-1)b\mathrm{d}x_2 \\ q^{-1}(\mathrm{d}c)x_2 + (q^{-2}-1)d\mathrm{d}x_1 & q^{-2}(\mathrm{d}d)x_2 + (q^{-2}-1)d\mathrm{d}x_2 \end{pmatrix}$$

$$\mathrm{d}x_1 \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = - \begin{pmatrix} \mathrm{d}a\mathrm{d}x_1 & q^{-1}\mathrm{d}b\mathrm{d}x_1 \\ \mathrm{d}c\mathrm{d}x_1 & q^{-1}\mathrm{d}d\mathrm{d}x_1 \end{pmatrix}$$

$$\mathrm{d}x_2 \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = - \begin{pmatrix} q^{-1}\mathrm{d}a\mathrm{d}x_2 + (1-q^{-2})\mathrm{d}b\mathrm{d}x_1 & \mathrm{d}b\mathrm{d}x_2 \\ q^{-1}\mathrm{d}c\mathrm{d}x_2 + (1-q^{-2})\mathrm{d}d\mathrm{d}x_1 & \mathrm{d}d\mathrm{d}x_2 \end{pmatrix}$$

$$\Delta x_i = 1 \otimes x_i + \Delta_R(x_i), \quad \Delta_*(\mathrm{d}x_i) = 1 \otimes \mathrm{d}x_i + \Delta_{R*}(\mathrm{d}x_i)$$

*along with the coproduct of* $\Omega(\mathbb{C}_q[GL_2])$ *a sub-super Hopf algebra. Moreover, the canonical coaction* $\Delta_R : \mathbb{C}_q^2 \to \mathbb{C}_q^2 \otimes \mathbb{C}_q[GL_2]{\ltimes\!\!\!\cdot}\mathbb{C}_q^2$ *is differentiable.*

*Proof.* We apply Theorem 6.3.3. The $\mathbb{C}_q[GL_2]$-crossed module structure has coaction $\Delta_R x_i = x_j \otimes t^j{}_i$ and right action $x_i \triangleleft t^k{}_l = q^{-1} x_j (R_{21}^{-1})^j{}_i{}^k{}_l$ which computes as shown. The rest follows similarly by computation for our choice of $R$ as in (6.4.1) with the $q$-Hecke normalisation $\alpha = 0$. The coaction of $\mathbb{C}_q[GL_2] \ltimes \mathbb{C}_q^2$ at the end is $\Delta_R x_i = 1 \otimes x_i + x_j \otimes t^j{}_i$. $\square$

**Remark 6.4.5.** Note that $\mathbb{C}_q[GL_2] \ltimes \mathbb{C}_q^2$ it a quantum deformation of a maximal parabolic $P \subset SL_3$ and one can check that this is indeed isomorphic to a quotient of $\mathbb{C}_q[SL_3]$. We have find its strongly bicovariant calculus and moreover it coacts differentiably on $\mathbb{C}_q^2$ by our results. The same construction still works when $q$ is a primitive $n$-th odd root of unity. Here we take $A = c_q[GL_2] = \mathbb{C}_q[GL_2]/(a^n - 1, b^n, c^n, d^n - 1)$ and $B = c_q^2 = \mathbb{C}_q^2/(x_1^n, x_2^n)$ with $c_q[GL_2] \ltimes c_q^2$ and strongly bicovariant exterior algebra as above with the additional relations $c_q[GL_2]$ and $c_q^2$.

# Chapter 7

# Exterior algebra on bicrossproduct Hopf algebras

Bicrossproduct Hopf algebras $A{\blacktriangleright\!\!\triangleleft}H$ were first introduced by Majid[23] in the search of self-dual algebraic structures. It still remains one of two main constructions known for quantum groups. Let $A, H$ be Hopf algebras forming a bicrossproduct and let $\Omega(A), \Omega(H)$ be their strongly bicovariant exterior algebras. In Section 7.1, we follow the similar philosophy of Chapters 5 and 6 to construct a super bicrossproduct $\Omega(A){\blacktriangleright\!\!\triangleleft}\Omega(H)$ and show that it gives a strongly bicovariant exterior algebra on $A{\blacktriangleright\!\!\triangleleft}H$ such that the canonical coaction $\Delta_R : H \to H \otimes A{\blacktriangleright\!\!\triangleleft}H$ is differentiable, i.e., extends to $\Delta_{R*} : \Omega(H) \to \Omega(H) \otimes \Omega(A{\blacktriangleright\!\!\triangleleft}H)$ as super comodule algebra. We note that differential calculi on finite group bicrossproducts $\mathbb{C}(M){\blacktriangleright\!\!\triangleleft}\mathbb{C}G$ where a finite group factorises into $G, M$ were classified in [40] but this did not address which of the calculi in the classification to take even in this case.

As an application, in section 7.2 we recover the natural exterior algebra of the Planck scale Hopf algebra $\mathbb{C}[g, g^{-1}]{\blacktriangleright\!\!\triangleleft}\mathbb{C}[p]$ in the known classification [37] such that it coacts differentiably on a copy $\mathbb{C}[r]$ of $\mathbb{C}[p]$. We also find the natural exterior algebra of the simplest Poincaré quantum group $\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}] = \mathbb{C}[SO_{1,1}]{\blacktriangleright\!\!\triangleleft}U(b_+)$ in a family of models of quantum spacetime[33] such that it coacts on $U(b_+)$ differentiably. Here $\mathbb{C}[SO_{1,1}]$ is the hyperbolic Hopf algebra and the $U(b_+)$ is regarded as the algebra of spacetime coordinates in $1+1$ dimensions.

## 7.1 Differentials by super bicrossproduct

Let $H$ and $A$ be two Hopf algebras with $A$ a left $H$-module algebra by left action $\triangleright$, and let $H$ be a right $A$-comodule coalgebra by coaction $\beta(h) = h^{\overline{(0)}} \otimes h^{\overline{(1)}}$. We suppose in this section that $\triangleright$ and $\Delta_R$ are compatible in a way [23] such that the cross product by $\triangleright$ and cross coproduct by $\Delta_R$ form a bicrossproduct Hopf algebra $A{\blacktriangleright\!\!\triangleleft}H$.

Let $\Omega(A)$ and $\Omega(H)$ be strongly bicovariant exterior algebras. Assume that $\Omega(A)$ is an $H$-covariant, i.e. $\Omega(A)$ is an $H$-module algebra by an action $\triangleright : H \otimes \Omega(A) \to \Omega(A)$ commuting with d and that this extends further to an action $\triangleright : \Omega(H)\underline{\otimes}\Omega(A) \to \Omega(A)$ differentiably such that

$$\mathrm{d}_A(\eta\triangleright\omega) = (\mathrm{d}_H\eta)\triangleright\omega + (-1)^{|\eta|}\eta\triangleright(\mathrm{d}_A\omega) \tag{7.1.1}$$

for all $\omega \in \Omega(A)$ and $\eta \in \Omega(H)$.

Dually, assume $\beta$ extends to a degree-preserving super coaction $\beta_* : \Omega(H) \to \Omega(H)\underline{\otimes}\Omega(A)$, denoted by $\beta_*(\eta) = \eta^{\overline{(0)}*} \otimes \eta^{\overline{(1)}*}$, such that $\Omega(H)$ is a super $\Omega(A)$-comodule coalgebra and

$$\beta_*(\mathrm{d}_H\eta) = \mathrm{d}_H\eta^{\overline{(0)}*} \otimes \eta^{\overline{(1)}*} + \eta^{\overline{(0)}*} \otimes \mathrm{d}_A\eta^{\overline{(1)}*}. \tag{7.1.2}$$

If $\triangleright$ and $\beta_*$ obey the super bicrossproduct conditions:

$$\epsilon(\eta\triangleright\omega) = \epsilon(\eta)\epsilon(\omega) \tag{7.1.3}$$

$$\Delta_*(\eta\triangleright\omega) = (-1)^{(|\eta_{(1)}{}^{\overline{(1)}*}|+|\eta_{(2)}|)|\omega_{(1)}|}\eta_{(1)}{}^{\overline{(0)}*}\triangleright\omega_{(1)} \otimes \eta_{(1)}{}^{\overline{(1)}*}(\eta_{(2)}\triangleright\omega_{(2)}) \tag{7.1.4}$$

$$\beta_*(\eta\xi) = (-1)^{(|\eta_{(1)}{}^{\overline{(1)}*}|+|\eta_{(2)}|)|\xi^{\overline{(0)}*}|}\eta_{(1)}{}^{\overline{(0)}*}\xi^{\overline{(0)}*} \otimes \eta_{(1)}{}^{\overline{(1)}*}(\eta_{(2)}\triangleright\xi^{\overline{(1)}*}) \tag{7.1.5}$$

$$(-1)^{|\omega||\eta_{(2)}{}^{\overline{(1)}*}|+|\eta_{(1)}||\eta_{(2)}{}^{\overline{(0)}*}|}\eta_{(2)}{}^{\overline{(0)}*} \otimes (\eta_{(1)}\triangleright\omega)\eta_{(2)}{}^{\overline{(1)}*} = \eta_{(1)}{}^{\overline{(0)}*} \otimes \eta_{(1)}{}^{\overline{(1)}*}(\eta_{(2)}\triangleright\omega) \tag{7.1.6}$$

then we have a bicrossproduct super Hopf algebra $\Omega(A){\blacktriangleright\!\!\triangleleft}\Omega(H)$ with product and coproduct

$$(\omega\otimes\eta)(\tau\otimes\xi)=(-1)^{|\eta_{(2)}||\tau|}\omega(\eta_{(1)}\triangleright\tau)\otimes\eta_{(2)}\xi$$

$$\Delta_*(\omega\otimes\eta)=(-1)^{|\omega_{(2)}||\eta_{(1)}{}^{\overline{(0)}*}|}\omega_{(1)}\otimes\eta_{(1)}{}^{\overline{(0)}*}\otimes\omega_{(2)}\eta_{(1)}{}^{\overline{(1)}*}\otimes\eta_{(2)}$$

for all $\omega,\tau\in\Omega(A)$, $\eta,\xi\in\Omega(H)$. We omit the proof since this close to the usual version [19, 23] just with some extra signs.

**Theorem 7.1.1.** *Let $A,H$ be Hopf algebras and form a bicrossproduct $A{\blacktriangleright\!\!\triangleleft}H$ and let $\Omega(A)$ and $\Omega(H)$ be strongly bicovariant exterior algebras with $\triangleright,\beta_*$ obey the conditions (7.1.1)-(7.1.6). Then $\Omega(A{\blacktriangleright\!\!\triangleleft}H):=\Omega(A){\blacktriangleright\!\!\triangleleft}\Omega(H)$ is a strongly bicovariant exterior algebra on $A{\blacktriangleright\!\!\triangleleft}H$ with differential*

$$\mathrm{d}(\omega\otimes\eta)=\mathrm{d}_A\omega\otimes\eta+(-1)^{|\omega|}\omega\otimes\mathrm{d}_H\eta.$$

*Proof.* This is clear since the product here is the left super cross product, which is a left reversal of the right-handed super cross product and the coproduct is the super right cross coproduct, which are similar to the proof of Theorem 6.2.1 with d the graded tensor product differential.

□

In practice, we typically only need to know that the bicrossproduct action $\triangleright$ and coaction $\beta$ extend to degree 1 to construct the super bicrossproduct, since the extension to higher degrees is determined. Note also that if $\Omega(A)$ is an $H$-module algebra and if $\Omega(H)$ is the maximal prolongation of $\Omega^1(H)$, then by the left-hand reversal of Lemma 5.1.4, $\Omega(A)$ is a super $\Omega(H)$-module algebra such that (7.1.1) holds.

**Lemma 7.1.2.** *Let $A,H$ be Hopf algebras forming a bicrossproduct $A{\blacktriangleright\!\!\triangleleft}H$. Let $\Omega(A)$ be a super left $\Omega(H)$-module algebra such that (7.1.1) holds, suppose that the coaction* *extends to a well-defined map* $\beta_*:\Omega^1(H)\to\Omega^1(H)\otimes A\oplus H\otimes\Omega^1(A)$ *by*

$$(1)\qquad \beta_*(h\mathrm{d}_H g)=h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})+h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A g^{\overline{(1)}})$$

*for all* $h,g\in H$*, and suppose that the action obeys* $\epsilon(\eta\triangleright\omega)=\epsilon(\eta)\epsilon(\omega)$ *and*

$$(2)\qquad \Delta_*(h\triangleright\mathrm{d}_A a)=h_{(1)}{}^{\overline{(0)}}\triangleright\mathrm{d}_A a_{(1)}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)})+h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A a_{(2)})$$

$$(3)\qquad h_{(2)}{}^{\overline{(0)}}\otimes(h_{(1)}\triangleright\mathrm{d}_A a)h_{(2)}{}^{\overline{(1)}}=h_{(1)}{}^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A a)$$

*for all* $h\in H$ *and* $a\in A$*. If* $\Omega(H)$ *is the maximal prolongation of* $\Omega^1(H)$*, and* $\Omega(A)$ *is the maximal prolongation of* $\Omega^1(A)$*, then* $\beta_*$ *extends to all degrees obeying (7.1.4)-(7.1.6) and form* $\Omega(A){\blacktriangleright\!\!\!\triangleleft}\Omega(H)$ *by Theorem 7.1.1.*

*Proof.* (i) First we first check that $\beta_*((\mathrm{d}_H h)g)$ also satisfies (7.1.5) for products from the other side,

$$\begin{aligned}\beta_*((\mathrm{d}_H h)g)=&\Delta_{R*}(\mathrm{d}_H(hg)-h\mathrm{d}_H g)\\ =&\mathrm{d}_H(hg)^{\overline{(0)}}\otimes(hg)^{\overline{(1)}}+(hg)^{\overline{(0)}}\otimes\mathrm{d}_A(hg)^{\overline{(1)}}-h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})\\ &-h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A g^{\overline{(1)}})\\ =&\mathrm{d}_H(h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}})\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})+h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes\mathrm{d}_A(h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}}))\\ &-h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})-h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A g^{\overline{(1)}})\\ =&(\mathrm{d}_H h_{(1)}{}^{\overline{(0)}})g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})+h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes(\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright g^{\overline{(1)}})\\ &+h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright g^{\overline{(1)}})\\ =&((\mathrm{d}_H h)_{(1)}{}^{\overline{(0)}})g^{\overline{(0)}}\otimes(\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h)_{(2)}\triangleright g^{\overline{(1)}}).\end{aligned}$$

Next, we prove that $\beta_*$ extends by (7.1.5) to $\Omega^2(H)$ for the maximal prolongation. Suppose $h\mathrm{d}_H g=0$ in $\Omega^1(H)$ (a sum of such terms understood) then $\beta_*(h\mathrm{d}_H g)=0$ tells us that

$$h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})=0,\quad h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A g^{\overline{(1)}})=0.$$

Applying $\mathrm{d}_H\otimes\mathrm{id}$ to the first equation, we have the following $\Omega^2(H)\otimes A$-part of $\beta_*(\mathrm{d}_H r\mathrm{d}_H s+\mathrm{d}_H b\mathrm{d}_H c)$

$$\mathrm{d}_H h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g^{\overline{(1)}})=0.$$

Applying $\mathrm{id}\otimes\mathrm{d}_A$ to the second equation, we have the following $H\otimes\Omega^2(A)$-part of $\beta_*(\mathrm{d}_H r\mathrm{d}_H s+\mathrm{d}_H b\mathrm{d}_H c)$

$$h_{(1)}{}^{\overline{(0)}}g^{\overline{(0)}}\otimes\big((\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright \mathrm{d}_A g^{\overline{(1)}})+h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright\mathrm{d}_A g^{\overline{(1)}})\big)=0.$$

Finally, applying $\mathrm{d}_H\otimes\mathrm{id}$ to the second equation and $\mathrm{id}\otimes\mathrm{d}_A$ to the first equation and subtracting them, we have the $\Omega^1(H)\underline{\otimes}\Omega^1(A)$-part of $\beta_*(\mathrm{d}_H r\mathrm{d}_H s+\mathrm{d}_H b\mathrm{d}_H c)$

$$\begin{aligned}&(\mathrm{d}_H h_{(1)}{}^{\overline{(0)}})g^{\overline{(0)}}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A g^{\overline{(1)}})\\&\qquad-h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g^{\overline{(0)}}\otimes((\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright g^{\overline{(1)}})+h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright g^{\overline{(1)}}))=0.\end{aligned}$$

Since $\triangleright:\Omega(H)\underline{\otimes}\Omega(A)\to\Omega(A)$ is defined and $\Omega(H)$ is the maximal prolongation of $\Omega^1(H)$, $\beta_*$ can be extended further to higher degree obeying (7.1.5).

(ii) Next we check that $\Delta_*((\mathrm{d}_H h)\triangleright a)$ satisfies (7.1.4)

$$\begin{aligned}\Delta_*((\mathrm{d}_H h)\triangleright a)&=\Delta_*(\mathrm{d}_A(h\triangleright a)-h\triangleright\mathrm{d}_A a)\\&=\mathrm{d}_A(h\triangleright a)_{(1)}\otimes(h\triangleright a)_{(2)}+(h\triangleright a)_{(1)}\otimes\mathrm{d}_A(h\triangleright a)_{(2)}-\Delta_*(h\triangleright\mathrm{d}_A a)\\&=\mathrm{d}_A(h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)})\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)})+(h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)})\otimes\mathrm{d}_A(h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)}))\\&\quad-h_{(1)}{}^{\overline{(0)}}\triangleright\mathrm{d}_A a_{(1)}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)})-h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright\mathrm{d}_A a_{(2)})\\&=\mathrm{d}_H h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)}\otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)})+(h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)})\otimes(\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright a_{(2)})\end{aligned}$$

$$+ h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright a_{(2)})$$
$$=(\mathrm{d}_H h)_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)} \otimes (\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h)_{(2)}\triangleright a_{(2)})$$

and one can find further that

$$\begin{aligned}\Delta_*((h\mathrm{d}_H g)\triangleright a) =&(h_{(1)}{}^{\overline{(0)}}\mathrm{d}_H g_{(1)}{}^{\overline{(0)}})\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g_{(1)}{}^{\overline{(1)}})((h_{(2)}g_{(2)})\triangleright a_{(2)})\\ &+ (h_{(1)}{}^{\overline{(0)}}g_{(1)}{}^{\overline{(0)}})\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A g_{(1)}{}^{\overline{(1)}})((h_{(2)}g_{(2)})\triangleright a_{(2)})\\ &+ (h_{(1)}{}^{\overline{(0)}}g_{(1)}{}^{\overline{(0)}})\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright g_{(1)}{}^{\overline{(1)}})((h_{(2)}\mathrm{d}_H g_{(2)})\triangleright a_{(2)})\end{aligned}$$

for all $a \in A$ as also required for (7.1.4). We can extends further to $\Delta_*((\mathrm{d}_H h)\triangleright \mathrm{d}_A a)$ and prove that it obeys (7.1.4) as follow

$$\begin{aligned}&\Delta_*((\mathrm{d}_H h)\triangleright \mathrm{d}_A a) = \Delta_*(\mathrm{d}_A(h\triangleright \mathrm{d}_A a))\\ &=\mathrm{d}_A(h\triangleright \mathrm{d}_A a)_{(1)} \otimes (h\triangleright \mathrm{d}_A a)_{(2)} + (-1)^{|(h\triangleright \mathrm{d}_A a)_{(1)}|}(h\triangleright \mathrm{d}_A a)_{(1)} \otimes \mathrm{d}_A(h\triangleright \mathrm{d}_A a)\\ &=\mathrm{d}_A(h_{(1)}{}^{\overline{(0)}}\triangleright \mathrm{d}_A a_{(1)}) \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)}) - h_{(1)}{}^{\overline{(0)}}\triangleright \mathrm{d}_A a_{(1)} \otimes \mathrm{d}_A(h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)}))\\ &\quad + \mathrm{d}_A(h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)}) \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A a_{(2)}) + h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)} \otimes \mathrm{d}_A(h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A a_{(2)}))\\ &=(\mathrm{d}_H h_{(1)}{}^{\overline{(0)}})\triangleright \mathrm{d}_A a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a_{(2)}) - h_{(1)}{}^{\overline{(0)}}\triangleright \mathrm{d}_A a_{(1)} \otimes (\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright a_{(2)})\\ &\quad - h_{(1)}{}^{\overline{(0)}}\triangleright \mathrm{d}_A a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright a_{(2)}) + (\mathrm{d}_H h_{(1)}{}^{\overline{(0)}})\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A a_{(2)})\\ &\quad + h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)} \otimes (\mathrm{d}_A h_{(1)}{}^{\overline{(1)}})(h_{(2)}\triangleright \mathrm{d}_A a_{(2)}) + h_{(1)}{}^{\overline{(0)}}\triangleright a_{(1)} \otimes h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright \mathrm{d}_A a_{(2)})\\ &=(-1)^{(|(\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}|+|(\mathrm{d}_H h)_{(2)}|)|(\mathrm{d}_A a)_{(1)}|}(\mathrm{d}_H h)_{(1)}{}^{\overline{(0)}}\triangleright(\mathrm{d}_A a)_{(1)} \otimes (\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h)_{(2)}\triangleright(\mathrm{d}_A a)_{(2)}).\end{aligned}$$

This then extends to all degrees since $\Delta_*$ of $\Omega(A)$ and $\triangleright : \Omega(H)\underline{\otimes}\Omega(A) \to \Omega(A)$ are defined by assumption and $\beta_* : \Omega(H) \to \Omega(H)\underline{\otimes}\Omega(A)$ is now defined since $\Omega(H)$ is the maximal prolongation of $\Omega^1(H)$.

(iii) Finally, we check that $\beta_*$ obeys (7.1.6). In fact by applying $\mathrm{d}_H \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d}_A$ to

$$h_{(2)}{}^{\overline{(0)}} \otimes (h_{(1)}\triangleright a)h_{(2)}{}^{\overline{(1)}} = h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a)$$

combined with assumption (3), we have

$$\begin{aligned}&\mathrm{d}_H h_{(2)}{}^{\overline{(0)}} \otimes (h_{(1)}\triangleright a)h_{(2)}{}^{\overline{(1)}} + h_{(2)}{}^{\overline{(0)}} \otimes ((\mathrm{d}_H h_{(1)})\triangleright a)h_{(2)}{}^{\overline{(1)}} + h_{(2)}{}^{\overline{(0)}} \otimes (h_{(1)}\triangleright a)\mathrm{d}_A h_{(2)}{}^{\overline{(1)}}\\&\quad= \mathrm{d}_H h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a) + h_{(1)}{}^{\overline{(0)}} \otimes \mathrm{d}_A h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright a) + h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}}(\mathrm{d}_H h_{(2)}\triangleright a)\end{aligned}$$

which is equivalent to

$$(\mathrm{d}_H h)_{(2)}{}^{\overline{(0)}} \otimes ((\mathrm{d}_H h)_{(1)}\triangleright a)(\mathrm{d}_H h)_{(2)}{}^{\overline{(1)}} = (\mathrm{d}_H h)_{(1)}{}^{\overline{(0)}} \otimes (\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h)_{(2)}\triangleright a).$$

Furthermore, one can find that

$$(h\mathrm{d}_H g)_{(2)}{}^{\overline{(0)}} \otimes ((h\mathrm{d}_H g)_{(1)}\triangleright a)(h\mathrm{d}_H g)_{(2)}{}^{\overline{(1)}} = (h\mathrm{d}_H g)_{(1)}{}^{\overline{(0)}} \otimes (h\mathrm{d}_H g)_{(1)}{}^{\overline{(1)}}((h\mathrm{d}_H g)_{(2)}\triangleright a).$$

We can also extend this by applying $\mathrm{d}_H \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d}_A$ to the assumption (3), where we have

$$\begin{aligned}&\mathrm{d}_H h_{(2)}{}^{\overline{(0)}} \otimes (h_{(1)}\triangleright \mathrm{d}_A a)h_{(2)}{}^{\overline{(1)}} + h_{(2)}{}^{\overline{(0)}} \otimes \big(((\mathrm{d}_H h_{(1)})\triangleright \mathrm{d}_A a)h_{(2)}{}^{\overline{(1)}} - (h_{(1)}\triangleright \mathrm{d}_A)\mathrm{d}_A h_{(2)}{}^{\overline{(1)}}\big)\\&\quad= \mathrm{d}_H h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}}(h_{(2)}\triangleright \mathrm{d}_A a) + h_{(1)}{}^{\overline{(1)}} \otimes \big((\mathrm{d}_A h_{(1)})(h_{(2)}\triangleright \mathrm{d}_A a) + h_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h_{(2)})\triangleright \mathrm{d}_A a)\big)\end{aligned}$$

which is equivalent to

$$\begin{aligned}&(-1)^{|(\mathrm{d}_H h)_{(2)}{}^{\overline{(1)}}||\mathrm{d}_A a|}(\mathrm{d}_H h)_{(2)}{}^{\overline{(0)}} \otimes ((\mathrm{d}_H h)_{(1)}\triangleright \mathrm{d}_A a)(\mathrm{d}_H h)_{(2)}{}^{\overline{(1)}}\\&\qquad\qquad= (\mathrm{d}_H h)_{(1)}{}^{\overline{(0)}} \otimes (\mathrm{d}_H h)_{(1)}{}^{\overline{(1)}}((\mathrm{d}_H h)_{(2)}\triangleright \mathrm{d}_A).\end{aligned}$$

Since $\triangleright$ and $\beta_*$ are defined for $\Omega(H)$, one can extend the above equation further to higher degree by applying $\mathrm{d}_H \otimes \mathrm{id} + (-1)^{|\ |}\mathrm{id} \otimes \mathrm{d}_A$ to the lower degree equation, and thus $\beta_*$ obeys (7.1.6). This completes the proof. □

This lemma assists with the data for Theorem 7.1.1. Moreover, as part of theory of bicrossproduct Hopf algebras, there is a covariant right coaction $\Delta_R : H \to H \otimes A{\blacktriangleright\!\!\triangleleft} H$

given by $\Delta_R h = h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}} \otimes h_{(2)}$. The following proposition shows that this coaction is differentiable.

**Corollary 7.1.3.** *If $\triangleright$ and $\beta_*$ obey the condition on Theorem 7.1.1 then $\Delta_R : H \to H \otimes A{\blacktriangleright\!\!\triangleleft} H$ as above is differentiable.*

*Proof.* Since the coaction $\Omega(H) \to \Omega(H)\underline{\otimes}\Omega(A)$ and action $\Omega(H)\underline{\otimes}\Omega(A) \to \Omega(A)$ globally exist as assumed in Theorem 6.2.1, it is clear that $\Delta_{R*}\eta = \eta_{(1)}{}^{\overline{(0)}*} \otimes \eta_{(1)}{}^{\overline{(1)}*} \otimes \eta_{(2)}$ is well-defined and gives a coaction of $\Omega(A{\blacktriangleright\!\!\triangleleft} H)$ on $\Omega(H)$. For example, on degree 1 we have

$$\Delta_{R*}(\mathrm{d}_H h) = \mathrm{d}_H h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}} \otimes h_{(2)} + h_{(1)}{}^{\overline{(0)}} \otimes \mathrm{d}_A h_{(1)}{}^{\overline{(1)}} \otimes h_{(2)} + h_{(1)}{}^{\overline{(0)}} \otimes h_{(1)}{}^{\overline{(1)}} \otimes \mathrm{d}_H h_{(2)}.$$

□

## 7.2 Examples of exterior algebras from super bicrossproduct

We now turn to some examples of exterior algebras constructed as super bicrossproducts. Our warm-up example is the 'Planck scale' Hopf algebra $\mathbb{C}[g, g^{-1}]{\blacktriangleright\!\!\triangleleft}\mathbb{C}[p]$ generated by $g, g^{-1}, p$ with

$$[p, g] = \lambda(1-g)g, \quad \Delta g = g \otimes g, \quad \Delta p = 1 \otimes p + p \otimes g$$

where $\lambda = \imath\frac{\hbar}{G}$ is an imaginary constant built from Planck's constant and the gravitational constant $G$, and $g = e^{-\frac{x}{G}}$ where $x$ is the spatial position in [22] (while we work algebraically with $g$ as a generator). This turned out to be a Drinfeld twist of $U(b_+)$ and we will obtain the same calculus as found in [37] by twisting methods. Let $\mathbb{C}[r]$ be another copy of $\mathbb{C}[p]$ and let $\Omega(\mathbb{C}[r])$ be an exterior algebra with

$$[r, \mathrm{d}r] = \lambda \mathrm{d}r, \quad (\mathrm{d}r)^2 = 0.$$

Also as part of the theory of bicrossproducts there is a natural coaction $\Delta_R : \mathbb{C}[r] \to \mathbb{C}[r] \otimes \mathbb{C}[g, g^{-1}]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]$ given by $\Delta_R r = 1 \otimes p + r \otimes g$ making $\mathbb{C}[r]$ a comodule-algebra. This coaction extends to $\Omega(\mathbb{C}[r])$ by $\Delta_R \mathrm{d}r = \mathrm{d}r \otimes g$.

**Proposition 7.2.1.** *There is a unique strongly bicovariant exterior algebra* $\Omega(\mathbb{C}[g, g^{-1}]\blacktriangleright\!\!\triangleleft\mathbb{C}[p])$ *such that* $\Delta_R$ *is differentiable. This has relations*

$$[\mathrm{d}g, g] = 0, \quad [p, \mathrm{d}p] = \lambda \mathrm{d}p, \quad [\mathrm{d}p, g] = \lambda g \mathrm{d}g, \quad [\mathrm{d}g, p] = \lambda(1-g)\mathrm{d}g$$

$$(\mathrm{d}p)^2 = (\mathrm{d}g)^2 = 0, \quad \mathrm{d}p\mathrm{d}g = -\mathrm{d}g\mathrm{d}p$$

*Furthermore,* $\Omega(\mathbb{C}[g, g^{-1}]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]) = \Omega(\mathbb{C}[g, g^{-1}])\blacktriangleright\!\!\triangleleft\Omega(\mathbb{C}[p])$ *with coproduct*

$$\Delta g = g \otimes g, \quad \Delta_* \mathrm{d}g = \mathrm{d}g \otimes g + g \otimes \mathrm{d}g$$

$$\Delta p = 1 \otimes p + p \otimes g, \quad \Delta_* \mathrm{d}p = 1 \otimes \mathrm{d}p + \mathrm{d}p \otimes g + p \otimes \mathrm{d}g.$$

*Proof.* If it exists then $\Delta_{R*}\mathrm{d}r = \mathrm{d}r \otimes g + 1 \otimes \mathrm{d}p + r \otimes \mathrm{d}g$. For this to extend in a well-defined way to $\Omega^1(\mathbb{C}[p])$ as in (5.1.2) we require $\Delta_{R*}(\mathrm{d}r.r - r\mathrm{d}r + \lambda \mathrm{d}r) = 0$ which gives $[p, \mathrm{d}p] = \lambda \mathrm{d}p$, $[\mathrm{d}g, g] = 0$ and

$$[\mathrm{d}g, p] + [\mathrm{d}p, g] + \lambda \mathrm{d}g = 0.$$

By requiring $\Delta_{R*}(\mathrm{d}r)^2 = 0$ we find that $[g, \mathrm{d}p] = \lambda g \mathrm{d}g$ which implies $[\mathrm{d}g, p]$ as stated. We also find the relations on degree 2 as stated. One can check that they obey Leibniz rule, making $\Omega(\mathbb{C}[g, g^{-1}]\blacktriangleright\!\!\triangleleft\mathbb{C}[p])$ an exterior algebra.

Furthermore, one can find that $\Omega(\mathbb{C}[p])$ is a super right $\Omega(\mathbb{C}[g, g^{-1}])$-comodule and $\Omega(\mathbb{C}[g, g^{-1}])$ is a super right $\Omega(H)$-module by the following actions and coactions

$$p{\triangleright}g = \lambda(1-g)g, \quad \mathrm{d}p{\triangleright}g = -\lambda g \mathrm{d}g, \quad p{\triangleright}\mathrm{d}g = \lambda(1-g)\mathrm{d}g, \quad \mathrm{d}p{\triangleright}\mathrm{d}g = 0$$

$$\beta(p) = p \otimes g, \quad \beta_*(\mathrm{d}p) = \mathrm{d}p \otimes g + p \otimes \mathrm{d}g.$$

It is a routine calculation to show that the stated action and coaction obey the condition on Theorem 7.1.1. Therefore super bicrossproduct $\Omega(\mathbb{C}[g,g^{-1}])\blacktriangleright\!\!\!\triangleleft\Omega(\mathbb{C}[p])$ gives the same exterior algebra as $\Omega(\mathbb{C}[g,g^{-1}]\blacktriangleright\!\!\!\triangleleft\mathbb{C}[p])$ and with the stated coproduct. □

The next most complicated example in this context is the quantum Poincaré group $\mathbb{C}[\mathrm{Poinc}_{1,1}] = \mathbb{C}[SO_{1,1}]\blacktriangleright\!\!\!\triangleleft U(\mathbb{R}\rtimes\mathbb{R})$ for which we follow the construction in [33]. Here $\mathbb{C}[SO_{1,1}]$ is the 'hyperbolic Hopf algebra' generated by $c = \cosh\alpha$ and $s = \sinh\alpha$ with relations $c^2 - s^2 = 1$ and $cs = sc$. (We work algebraically with $c, s$ and do not need $\alpha$ itself.) This is a Hopf algebra with

$$\Delta\begin{pmatrix} c & s \\ s & c\end{pmatrix} = \begin{pmatrix} c & s \\ s & c\end{pmatrix}\otimes\begin{pmatrix} c & s \\ s & c\end{pmatrix},\quad \epsilon\begin{pmatrix} c & s \\ s & c\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\quad S\begin{pmatrix} c & s \\ s & c\end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c\end{pmatrix}.$$

Also recall that $U(\mathbb{R}\rtimes\mathbb{R})$ is a Hopf algebra generated by $a_0$ and $a_1$ with relation $[a_0, a_1] = \lambda a_0$ for some $\lambda \in \mathbb{C}$, and primitive comultiplication $\Delta a_i = 1\otimes a_i + a_i \otimes 1$ for $i = 0, 1$. $\mathbb{C}[SO_{1,1}]$ coacts on $U(\mathbb{R}\rtimes\mathbb{R})$ and $U(\mathbb{R}\rtimes\mathbb{R})$ acts on $\mathbb{C}[SO_{1,1}]$ by

$$a_0\triangleright\begin{pmatrix} c \\ s\end{pmatrix} = \lambda\begin{pmatrix} s^2 \\ sc\end{pmatrix},\quad a_1\triangleright\begin{pmatrix} c \\ s\end{pmatrix} = \lambda\begin{pmatrix} cs - s \\ c^2 - c\end{pmatrix},\quad \Delta_R\begin{pmatrix} a_0 & a_1\end{pmatrix} = \Delta_R\begin{pmatrix} a_0 & a_1\end{pmatrix}\otimes\begin{pmatrix} c & s \\ s & c.\end{pmatrix}$$

Thus their bicrossproduct $\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}] = \mathbb{C}[SO_{1,1}]\blacktriangleright\!\!\!\triangleleft U(\mathbb{R}\rtimes\mathbb{R})$ contains $\mathbb{C}[SO_{1,1}]$ as sub-Hopf algebra, $U(\mathbb{R}\rtimes\mathbb{R})$ as subalgebra, and with the following additional cross-relations and coproducts[33]

$$[a_0, \begin{pmatrix} c \\ s\end{pmatrix}] = \lambda s\begin{pmatrix} s \\ c\end{pmatrix},\quad [a_1, \begin{pmatrix} c \\ s\end{pmatrix}] = \lambda(c-1)\begin{pmatrix} s \\ c\end{pmatrix},\quad \Delta a_i = 1\otimes a_i + \Delta_R a_i.$$

Finally, we let $U(\mathbb{R}\rtimes\mathbb{R})$ be another copy with generators $t, x$ in place of $a_0, a_1$ with exterior algebra[43]

$$[\mathrm{d}x, x] = \lambda\theta',\quad [\mathrm{d}x, t] = 0,\quad [\mathrm{d}t, x] = \lambda\mathrm{d}x,\quad [\mathrm{d}t, t] = \lambda(\mathrm{d}t - \theta'),\quad [\theta', x] = 0,\quad [\theta', t] = -\lambda\theta'$$

$$\mathrm{d}\theta' = 0, \quad (\mathrm{d}x)^2 = (\mathrm{d}t)^2 = 0, \quad \mathrm{d}x\mathrm{d}t = -\mathrm{d}t\mathrm{d}x, \quad \mathrm{d}x.\theta' = -\theta'\mathrm{d}x, \quad \mathrm{d}t.\theta' = -\theta'\mathrm{d}t.$$

It is known that this is covariant under $\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}]$ by coaction

$$\Delta_R : U(\mathbb{R}\bowtie\!\!\!\!\lhd\mathbb{R}) \to U(\mathbb{R}\bowtie\!\!\!\!\lhd\mathbb{R}) \otimes \mathbb{C}[\mathrm{Poinc}_{1,1}], \quad \Delta_R(t \quad x) = 1 \otimes (a_0 \quad a_1) + (t \quad x) \otimes \begin{pmatrix} c & s \\ s & c \end{pmatrix}$$

in the same matrix notation as for coproducts above.

**Proposition 7.2.2.** *There is a unique strongly bicovariant exterior algebra* $\Omega(\mathbb{C}[\mathrm{Poinc}_{1,1}])$ *such that the above coaction on* $U(\mathbb{R}\bowtie\!\!\!\!\lhd\mathbb{R})$ *is differentiable. It contains* $\Omega(U(\mathbb{R}\bowtie\!\!\!\!\lhd\mathbb{R}))$ *as sub-exterior algebras and has the additional relations on degree 1 and 2*

$$(\mathrm{d}c)s = s\mathrm{d}c, \quad (\mathrm{d}s)c = c\mathrm{d}s, \quad (\mathrm{d}c)c = c\mathrm{d}c = s\mathrm{d}s = (\mathrm{d}s)s$$

$$[\mathrm{d}a_0, \begin{pmatrix} c \\ s \end{pmatrix}] = \lambda c \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}, \quad [\mathrm{d}a_1, \begin{pmatrix} c \\ s \end{pmatrix}] = \lambda s \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}$$

$$[a_0, \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}] = \lambda s \begin{pmatrix} \mathrm{d}s \\ \mathrm{d}c \end{pmatrix}, \quad [a_1, \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}] = \lambda(c-1) \begin{pmatrix} \mathrm{d}s \\ \mathrm{d}c \end{pmatrix}, \quad [\theta', \begin{pmatrix} c \\ s \end{pmatrix}] = \lambda(c-1) \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}$$

$$\{\mathrm{d}a_0, \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}\} = \begin{pmatrix} 0 \\ \lambda\mathrm{d}s\mathrm{d}c \end{pmatrix}, \quad \{\mathrm{d}a_1, \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}\} = \begin{pmatrix} \lambda\mathrm{d}c\mathrm{d}s \\ 0 \end{pmatrix}.$$

*Furthermore,* $\Omega(\mathbb{C}[\mathrm{Poinc}_{1,1}]) = \Omega(\mathbb{C}[SO_{1,1}]){\blacktriangleright\!\!\!\lhd}\Omega(U(\mathbb{R}\bowtie\!\!\!\!\lhd\mathbb{R}))$ *with coproduct that of* $\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}]$ *on degree 0 and*

$$\Delta_*\mathrm{d}c = \mathrm{d}c \otimes c + \mathrm{d}s \otimes s + c \otimes \mathrm{d}c + s \otimes \mathrm{d}s, \quad \Delta\mathrm{d}s = \mathrm{d}c \otimes s + \mathrm{d}s \otimes c + c \otimes \mathrm{d}s + s \otimes \mathrm{d}c$$

$$\Delta_*\mathrm{d}a_0 = 1 \otimes \mathrm{d}a_0 + \mathrm{d}a_0 \otimes c + \mathrm{d}a_1 \otimes s + a_0 \otimes \mathrm{d}c + a_1 \otimes \mathrm{d}s$$

$$\Delta_*\mathrm{d}a_1 = 1 \otimes \mathrm{d}a_1 + \mathrm{d}a_0 \otimes s + \mathrm{d}a_1 \otimes c + a_0 \otimes \mathrm{d}s + a_1 \otimes \mathrm{d}c$$

$$\Delta_*\theta' = 1 \otimes \theta' + \theta' \otimes 1 - 1 \otimes \mathrm{d}a_0 - \mathrm{d}a_0 \otimes 1 + \Delta_*\mathrm{d}a_0.$$

*Proof.* If $\Delta_{R*}$ exists obeying (5.1.2), we will need

$$\Delta_{R*}\mathrm{d}t = 1\otimes \mathrm{d}a_0 + \mathrm{d}t\otimes c + \mathrm{d}x\otimes s + t\otimes \mathrm{d}c + x\otimes \mathrm{d}s$$

$$\Delta_{R*}\mathrm{d}x = 1\otimes \mathrm{d}a_1 + \mathrm{d}t\otimes s + \mathrm{d}x\otimes c + t\otimes \mathrm{d}s + x\otimes \mathrm{d}c$$

$$\Delta_{R*}\theta' = 1\otimes \theta' - 1\otimes \mathrm{d}a_0 + \theta'\otimes 1 - \mathrm{d}t\otimes 1 + \Delta_{R*}\mathrm{d}t$$

By requiring $\Delta_{R*}([\mathrm{d}t, x]) = \Delta_{R*}(\lambda \mathrm{d}x)$ and $\Delta_{R*}([\mathrm{d}x, t]) = 0$, we find

$$(\mathrm{d}s)c = c\mathrm{d}s, \quad (\mathrm{d}c)s = s\mathrm{d}c, \quad (\mathrm{d}c)c + (\mathrm{d}s)s - c\mathrm{d}c - s\mathrm{d}s = 0.$$

$$[\mathrm{d}a_1, c] = [a_0, \mathrm{d}s], \quad [\mathrm{d}a_1, s] - [a_0, \mathrm{d}c] + \lambda((\mathrm{d}s)s - c\mathrm{d}c) = 0$$

$$[a_1, \mathrm{d}c] = [\mathrm{d}a_0, s] - \lambda \mathrm{d}s, \quad [\mathrm{d}a_0, c] - [a_1, \mathrm{d}s] + \lambda((\mathrm{d}c)c - s\mathrm{d}s - \mathrm{d}c) = 0.$$

Additionally, from $\mathrm{d}(c^2 - s^2) = 0$ combined with the condition $\mathrm{d}c.c + \mathrm{d}s.s - c\mathrm{d}c - s\mathrm{d}s = 0$ above, we find that $(\mathrm{d}c)c = s\mathrm{d}s$ and $(\mathrm{d}s)s = c\mathrm{d}c$. But then by requiring $\Delta_{R*}([\mathrm{d}x, x]) = \Delta_{R*}(\lambda\theta')$ we also find that $(\mathrm{d}c)c = c\mathrm{d}c$ and $(\mathrm{d}s)s = s\mathrm{d}s$. Thus we have $(\mathrm{d}c)c = c\mathrm{d}c = s\mathrm{d}s = (\mathrm{d}s)s$. Also by requiring $\Delta_{R*}(\mathrm{d}x)^2 = 0$ and $\Delta_{R*}(\mathrm{d}t)^2 = 0$ one can find that

$$[\mathrm{d}a_0, \begin{pmatrix} c \\ s \end{pmatrix}] = \lambda c \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}, \quad [\mathrm{d}a_1, \begin{pmatrix} c \\ s \end{pmatrix}] = \lambda s \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}$$

leading to the stated bimodule relations. The relations involving $\theta'$ are obtained by using the other relations. One can also find that the stated relations on degree 2 hold and $\Omega(\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}])$ contains $\Omega(U(\mathbb{R}\bowtie\!\!\!\triangleleft\mathbb{R}))$ from the above calculation.

Furthermore, $\Omega(\mathbb{C}[SO_{1,1}])$ is a super $\Omega(U(\mathbb{R}\bowtie\!\!\!\triangleleft\mathbb{R}))$-module and $\Omega(U(\mathbb{R}\bowtie\!\!\!\triangleleft\mathbb{R}))$ is a super $\Omega(\mathbb{C}[SO_{1,1}])$-comodule with actions and coactions

$$\mathrm{d}a_0 \triangleright \begin{pmatrix} c \\ s \end{pmatrix} = \lambda c \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}, \quad \mathrm{d}a_1 \triangleright \begin{pmatrix} c \\ s \end{pmatrix} = \lambda s \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}$$

$$a_0 \triangleright \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix} = \lambda s \begin{pmatrix} \mathrm{d}s \\ \mathrm{d}c \end{pmatrix}, \quad a_1 \triangleright \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix} = \lambda(c-1) \begin{pmatrix} \mathrm{d}s \\ \mathrm{d}c \end{pmatrix}$$

$$\beta_*(\mathrm{d}a_0) = \mathrm{d}a_0 \otimes c + \mathrm{d}a_1 \otimes s + a_0 \otimes \mathrm{d}c + a_1 \otimes \mathrm{d}s$$

$$\beta_*(\mathrm{d}a_1) = \mathrm{d}a_0 \otimes s + \mathrm{d}a_1 \otimes c + a_0 \otimes \mathrm{d}s + a_1 \otimes \mathrm{d}c$$

$$\theta \triangleright \begin{pmatrix} c \\ s \end{pmatrix} = \lambda(c-1) \begin{pmatrix} \mathrm{d}c \\ \mathrm{d}s \end{pmatrix}, \quad \beta_*(\theta') = \theta' \otimes 1 - \mathrm{d}a_0 \otimes 1 + \beta_*(\mathrm{d}a_0)$$

and one can check that they obey the conditions for Theorem 7.1.1. Therefore we can apply the super bicrossproduct and identify $\Omega(\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}]) = \Omega(\mathbb{C}[\mathrm{SO}_{1,1}]) \blacktriangleright\!\!\triangleleft \Omega(\mathrm{U}(\mathrm{R} \rtimes \mathbb{R}))$ with coproducts as stated. $\square$

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